



HP Prime
Teacher Workshop
For Teachers of Secondary Mathematics

Foreword

This book was designed to be the basis for a three-day teacher professional development institute. The purpose of that institute is to give a solid though basic understanding of the HP Prime graphing calculator and its ecosystem to teachers of mathematics at the secondary level. For the purposes of the teacher professional development institute, the activities in this book are meant to be supplemented with several activities found in the HP Prime AP Statistics materials and the HP Prime AP Calculus materials. These latter activities are not included here to avoid redundancy in translating and printing these three books.

The activities presented here all focus on interesting mathematical problems. They contain step by step instructions on how to use HP Prime to explore and solve the mathematical problem at hand. The activities include teacher notes and an answer key; there may also be one or more ideas for extension provided. Wherever possible, the activities involve new representations or new ways of looking at old representations. We want teachers to experience these activities just as their students will.

Some of the activities are “teacher activities” rather than student activities. While teachers may share such an activity with students, the activity was written with a teacher audience in mind. For these teacher activities, no answer key is needed (or provided).

The focus of all of the activities is to get teachers thinking about the technological experiences they want students to have. What are the key technological experiences most likely to help students gain a conceptual understanding of mathematics as well as procedural fluency and efficiency? These technological experiences center on multiple representations, dynamic links between representations, and representational fluency in general. The HP Prime graphing calculator and its ecosystem have been uniquely designed to provide many of these technological experiences, and to provide them in a way that is easy for the teacher to reproduce and engaging for the student.

Finally, I want to thank Mark Howell, Michael Grasse, Sharon Taylor, Eduardo Basurto, Eduardo Mancera, Amy Yonashiro and everyone else who contributed activity ideas or suggestions for improving these materials.

GT Springer
May 31, 2017

Table of Contents










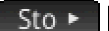


Activity Title	Page
Getting Acquainted with HP Prime	4
HP Prime Apps and Their Views	9
Introduction to Linear Functions	10
A Bridge to Quadratic Functions	13
The Intersection of Two Curves	18
An Infinite Series	22
Solving Trigonometric Equations	26
A Special System of Linear Equations	30
Identities, Equivalence, and Conditions	32
Operating on Functions	35
Mathematical Art	40
Peculiar Parabolas	44
Sicherman Dice	47
Conjecture and Proof: Midpoint Quadrilaterals	50
Cyclic and Bicentric Quadrilaterals	57
A Nautilus Shell	62
Pascal's Triangle	65
Rotation of Axes	68
Graphs Polar and Cartesian	72


Getting Acquainted With HP Prime

HP Prime is a color, touchscreen graphing calculator, with multi-touch capability, a Computer Algebra System (CAS), and a set of apps for exploring mathematical concepts and solving problems. For example, there is an Advanced Graphing app that lets you graph any relation in two variables (graphing something like $\sin(xy) = \cos(xy)$ for example), and a set of three apps for statistics (Statistics 1Var, Statistics 2Var, and Inference). In this section, we'll take a look at how to find your way around HP Prime, and get acquainted with the HP Prime app structure.

Document Conventions





First, here are a few conventions we'll use in this document:




- A key that initiates an un-shifted function is represented by an image of that key:
, ,  and so on.
- A key combination that initiates a shifted function (or inserts a character) is represented by the appropriate shift key ( or ) followed by the key for that function or character:
 initiates the natural exponential function and  inserts the letter F.
- The name of the shifted function may also be given in parentheses after the key combination:
 (Clear),  (Plot Setup)
- A key pressed to insert a digit is represented by that digit:
5, 7, 8, and so on.
- All fixed on-screen text—such as screen and field names—appear in bold:
CAS Settings, **Xstep**, **Decimal Mark**, and so on.
- A menu key selected by touching the screen is represented by an image of that key:
, , , and so on.




NOTE: You must use your finger or a capacitive stylus to tap a menu key or to select a menu item; as an alternative, you can often navigate to a selection and press .




- Cursor keys are represented by , , , and . You use these keys to move from field to field on a screen, or from one option to another in a list of options.

The ON-OFF key is at the bottom left of the keyboard. When a new HP Prime is turned on for the first time, a "splash" screen appears that invites the user to select a language and to make some initial setup choices. For most users, accepting the default options is the way to go.








The screen brightness can be increased by pressing and holding  and  or decreased by pressing and holding  and .

Take a minute to look at the layout of the keyboard. The top group of keys, with the black background, is primarily for navigating from one environment to another. Pressing  takes you to the home calculation screen, and pressing  takes you to a similar calculation environment for doing symbolic or exact computations. Pressing  takes you to a menu

where you can select from all the applications in the HP Prime, like **Advanced Graphing** or **Function**. The bottom group of keys is mainly for entering or editing mathematical expressions. There are also environments for entering lists, , matrices, , and user programs, .

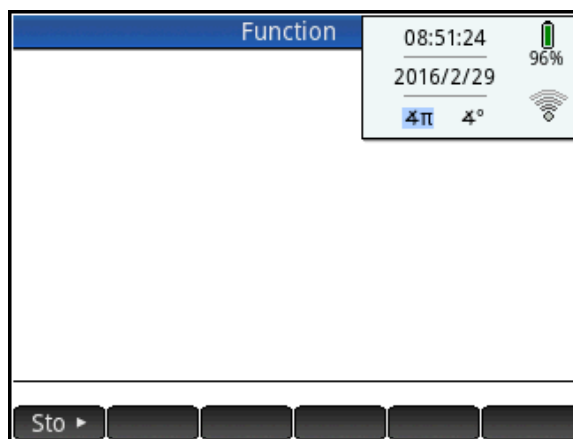
Some care was taken when deciding where to place certain keys. The number π for instance, is . The list delimiters, {}, appear just to the right of the LIST key,  and the matrix delimiters, [], appear just to the right of the MATRIX key, .

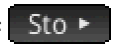
Things you can do in both CAS and Home views:

- Tap an item to select it or tap twice to copy it to the command line editor
- Tap and drag up or down to scroll through the history of calculations
- Press  to retrieve a previous entry or result from the other view
- Press the Toolbox key () to see the Math and CAS menus as well as the Catalog
- Press  to open a menu of easy-to-use templates
- Press  to exit these menus without making a selection
- Tap , , and  menu buttons

Home View

Turn on your HP Prime and take a look at the different sections of the screen in the **HOME** view. The top banner across the top is called the Title Bar, and it tells you what operating environment you are currently working in (like **HOME** or **FUNCTION SYMBOLIC VIEW**). If you press a shift key, an annunciator comes on at the left of the Title Bar. On the right, you see a battery level indicator, a clock, and the current angle mode. You can tap this Quick Settings section at the top right to see a calendar (by tapping the date and time), connect to a wireless classroom network (by tapping the wireless icon), or change the angle mode (by tapping the angle mode indicator).



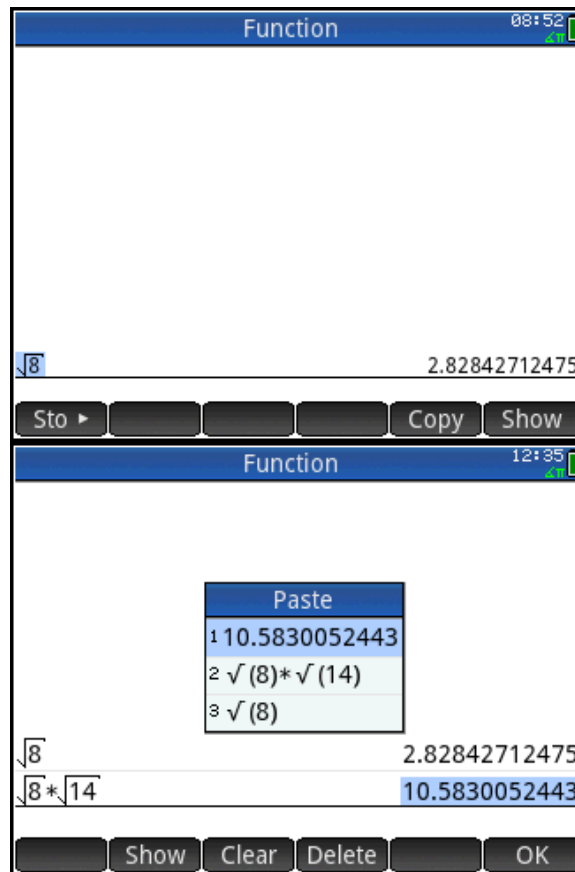
The middle section of the **HOME** view contains a history of past calculations. You can navigate through the history with the cursor keys, or using your finger to select (by tapping) or scroll (by swiping). The edit line is just below the history section. This is where you enter mathematical expressions to evaluate numerically. At the bottom are the menu keys, consisting of  in the **HOME** view. These menu keys are context sensitive. Their labels and function change depending on the environment you're in.

Home view is for numerical calculations.

1. Press **On** to turn on your HP Prime
2. Press **Settings** to see the Home view.
3. Press **Shift** **x²** **()** **8** **Enter** to evaluate $\sqrt{8}$.
4. Now double-tap $\sqrt{8}$ in the Home history to copy it back to the command line. Yes, HP Prime has a touchscreen!
5. Now we will multiply $\sqrt{8} \cdot \sqrt{14}$. Type **x** **Shift** **x²** **1** **Program** **4** **Matrix** **U** **Enter**.
6. Tap on the last result to select it and press **Shift** **View Copy** (Copy). You just copied the 10.583... to the Prime clipboard. Now you can paste it anywhere by pressing **Shift** **Menu Paste** (Paste). You can see the clipboard also contains our last few entries.
7. Press **Esc** to close the clipboard without pasting anything.

In Home view, all results evaluate to a real or a complex number, or a matrix, list, etc. of real and/or complex numbers. You can tap on any previous input or result in the history to select it. When you do, two new menu buttons appear:

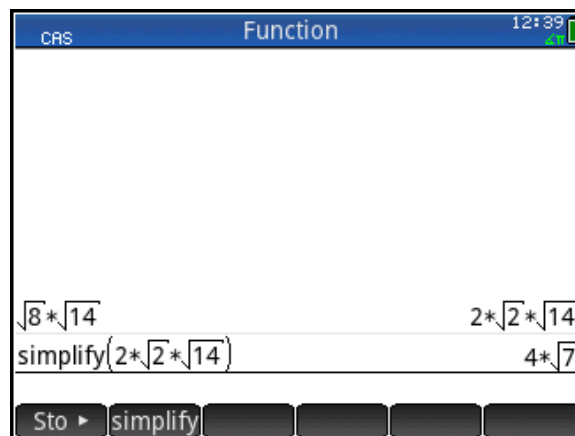
Copy and **Show**. The former copies the selection to the command line at the cursor position while the later typesets the selection in textbook format in full-screen mode.






CAS View






CAS view, on the other hand, is for symbolic or exact numerical results. Let's repeat our first calculation in CAS view.

1. Press **CAS** to open CAS view.
2. Press **Shift** **Menu Paste** to open the clipboard. Tap on the entry $\sqrt{8} \cdot \sqrt{14}$ to paste it into the CAS command line and press **Enter** to see the result.
3. Tap **Simplify** to simplify the results (depends on CAS Settings).








Home view returns numerical approximations while CAS always tries to return exact or symbolic results. It's up to you which view to use. In either Home or CAS views, you can double-tap an entry or result in history to copy it back to the command line. You can drag or swipe to scroll the history. Tap an entry and press  to delete it or press   (Clear) to clear the entire history.


Copy and Paste

As mentioned earlier, both the CAS and Home view histories use  to copy the selection to the command line. There is also   (Copy) and   (Paste) that copies the selection to the Prime clipboard and pastes from that clipboard to the cursor position. This functionality makes it possible to copy and paste from one environment to another anywhere in your HP Prime. With data, you can tap and hold, then drag to select a rectangular array of cells, then copy and paste anywhere else. Finally, with the HP Prime Virtual Calculator you can use the computer clipboard to copy an array of cells on a website or in a spreadsheet on your PC and paste the numerical data anywhere in the Prime Virtual Calculator. You can then send the data to another HP Prime.






Delete and Clear

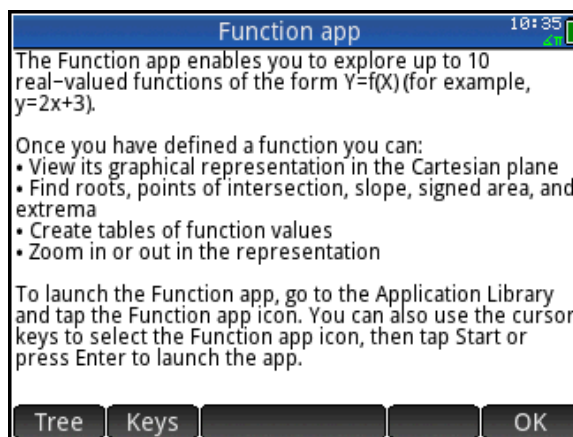
In both CAS and Home views, you can select an item in history and press  to delete it. Press   (Clear) to delete the entire history. If you are in any view (Symbolic, Plot Setup, etc.),   (Clear) will reset (clear) all settings in the current page of the view back to their factory defaults.

Help is Always Here!

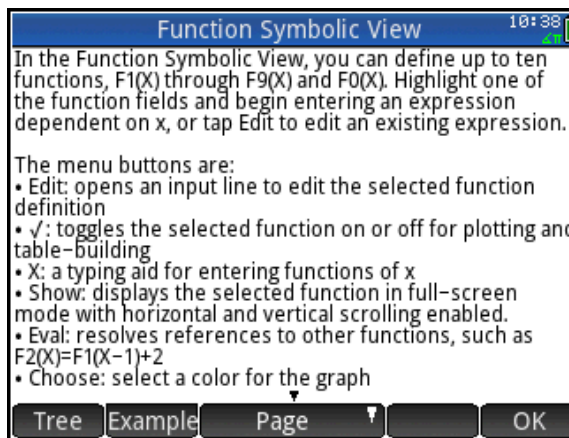
HP Prime includes a Help key right on the keyboard: . Pressing this key displays helpful information about any app as well as its major views. You can also get help on any catalog/editor (program, matrix, etc.) and any command or function. This section introduces you to the Help System.

Help on Apps and their Views

1. Press  and use the cursor pad to select an app (eg. Function); then press . The help page for the app is displayed. Use your finger to scroll the help page. Tap  when you are done. There are 2 additional menu keys:
 - : opens the entire help tree
 - : tap here, then press any key on the keyboard to get help on that key

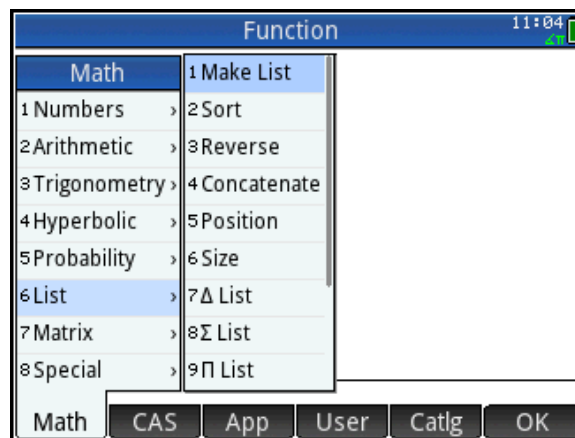


2. Tap an app (eg. Function) to open it; then press . The help page for the current view (Symbolic, Plot, Numeric, etc.) is displayed. The purpose of the view is described and the various menu keys are listed. Tap when you are done.



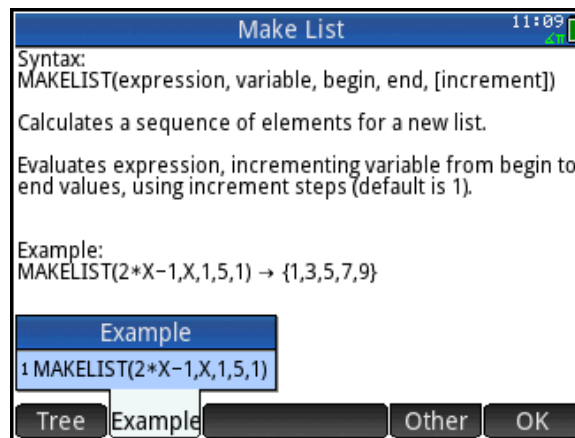
Help on Commands

3. Press to open the Toolbox menu. Tap to open the menu of common mathematical functions. Tap **List** to open the List submenu; the first entry is **Make List**. With **Make List** selected, press .
4. The help page for the MAKELIST command is displayed. The syntax is given, followed by a description of the purpose and use of the command. This is followed by one or more examples, as appropriate.
5. Tap and select an example to paste into the command line. In this case, the MAKELIST command will generate the list {1, 3, 5, 7, 9}.
6. Tap to see any related commands and open their help pages.



Exploring the Help Tree

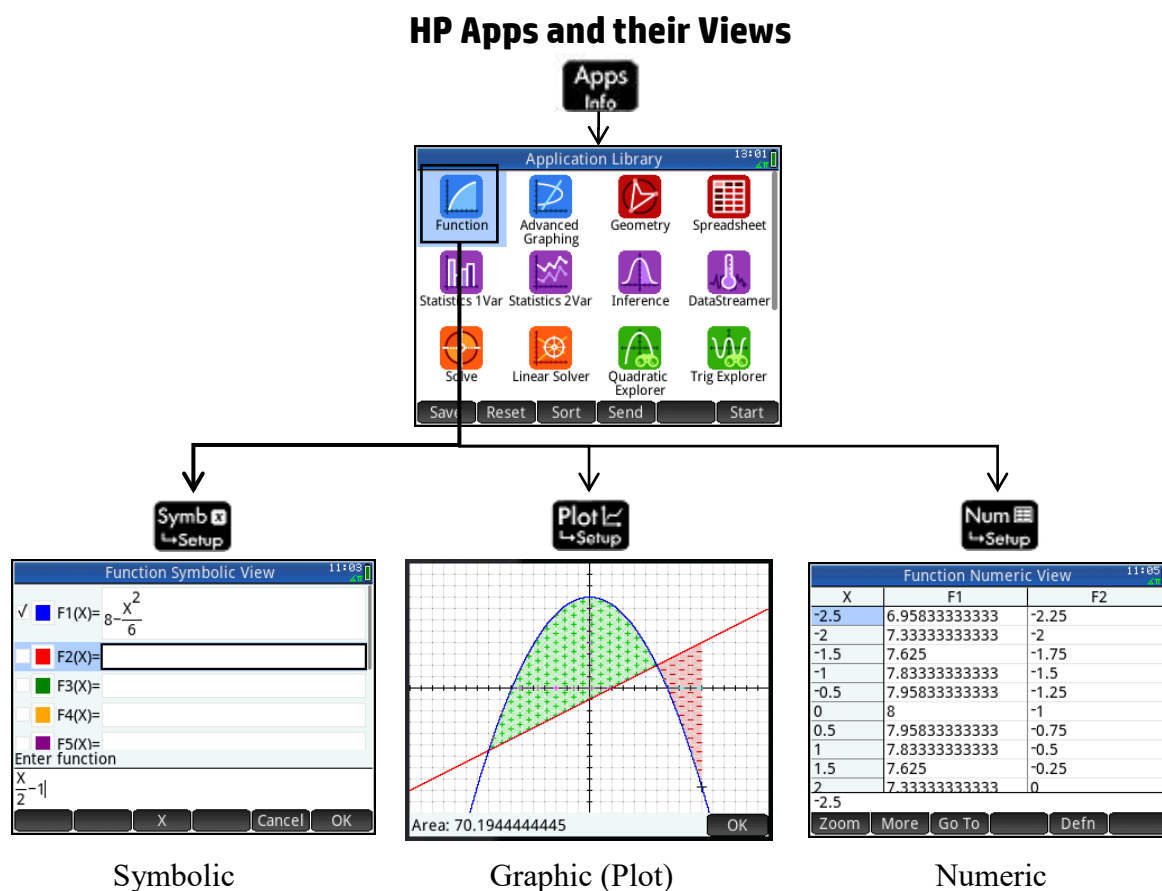
1. Tap to see the entire Help System topics tree. Click on the + box next to an item to expand it. Tap on an item to select it and then tap to view its page.
2. Tap and then press one of the keyboard keys to get help on that key.
3. Tap and enter a keyword. HP Prime will take you to an item in the tree containing that keyword.



HP Prime has an extensive on-board help system to get you started and to support you as you explore mathematical concepts and solve problems.

HP Prime Apps and Their Views

HP Prime uses an app-based architecture to reduce complexity. All HP Prime apps have a similar structure, utilizing the **Symb**, **Plot**, and **Num** keys to maintain simplicity and enhance learnability. These keys open the Symbolic, Plot, and Numeric views that correspond to the symbolic, graphic, and numeric forms of mathematical representations. The Prime app schema is shown in the figure below.



Press **Apps Info** to open the App Library. Tap on an app to start it, or navigate the library using the cursor pad and tap **Start** to launch the app.

Fill the app with data while you work; you can come back to this app anytime—all your data is automatically saved! You can save an app with a name you'll remember; then reset the original app and use it for something else. HP Apps have app functions as well as app variables; you can use them while in the app, or from the CAS view, Home view, or in programs.

Introduction to Linear Functions

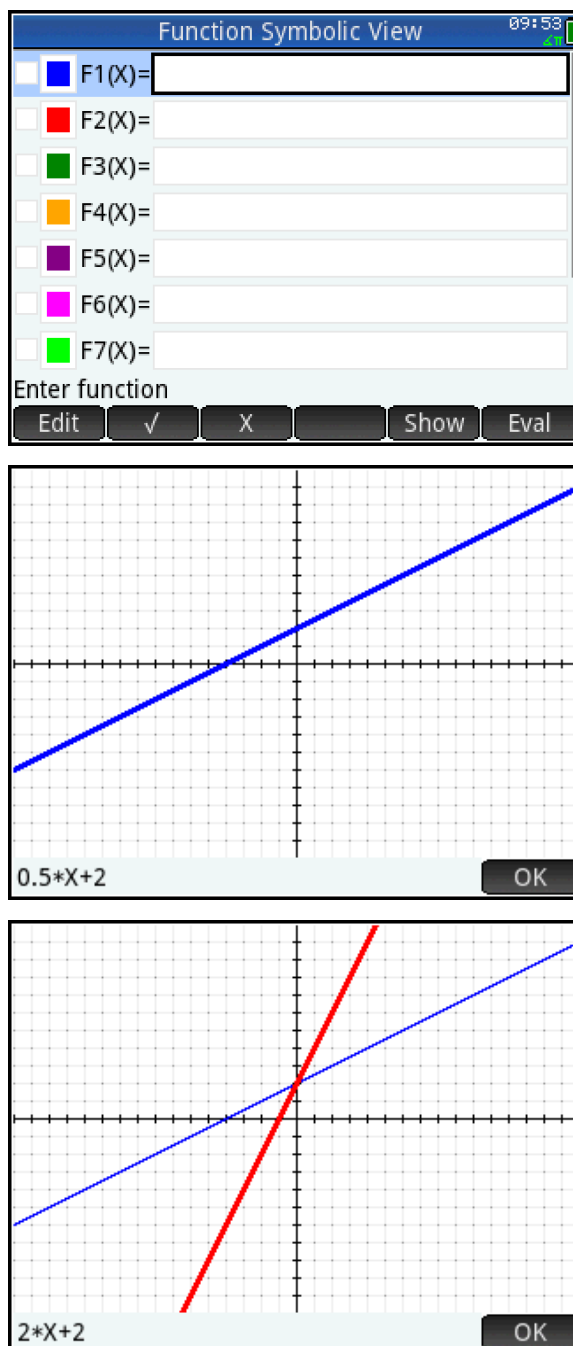
Points in the plane can be expressed as ordered pairs (x, y) . If a line is drawn through 2 or more points in the plane, then the line can be expressed symbolically as an equation in x and/or y . In this activity, you will learn about these linear equations using HP Prime.

1. Press **Apps** and tap on the **Function** app icon. The app opens in Symbolic view. Press **Symb** to return to this view at any time. Here you can enter functions and choose a color for plotting each function.
2. Press **Plot** to open Plot view. You will see a message that there are no functions to plot; just tap **OK** to continue.
3. Tap **Menu** to open the Plot view menu. Tap **Sketch** and sketch a line with your finger that intersects the y -axis at $y=2$, as in the figure to the right.

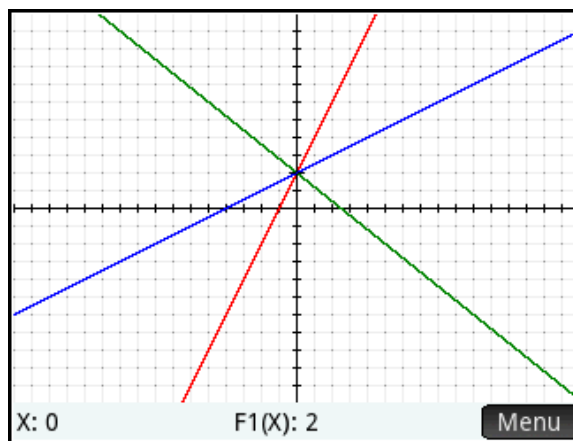
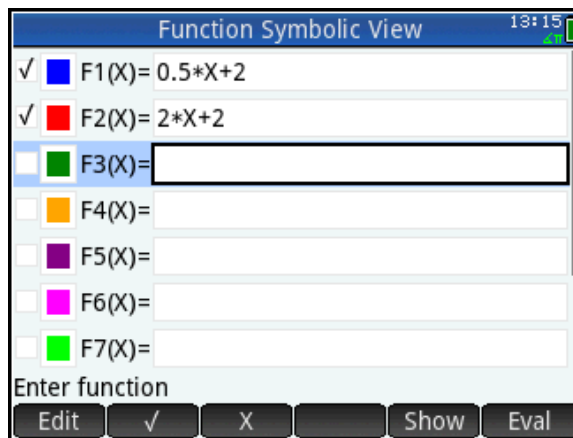
The expression for your line is displayed at the bottom left. So the equation of the line in the figure to the right is actually $Y=0.5*X+2$.

4. Tap **OK** to keep your line and record its equation in Symbolic view.
5. Now draw another line that intersects the y -axis at the same point as your first line. If your first attempt is not successful, just draw a new line with your finger. It will replace the first one.

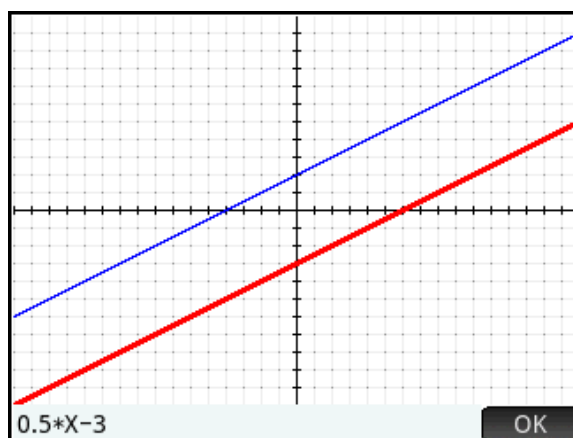
What do you notice about the equations of these two lines? What is the same about them and what is different?



6. Tap **OK** once to record your linear equation. Then tap **OK** again to leave Sketch mode.
7. Press **Symb** to return to Symbolic view. The equations of our two lines are both shown here.
8. In F3(X), record a new expression whose graph will be a line that intersects the y-axis at the same point as the other two lines.
9. Press **Plot** to see if you are correct!
10. Record the equations of your three lines below. Remember to use Y= for each one instead of F1(X)=, etc. For example, the first equation in the screen shot to the right above would be recorded as $Y=0.5*X+2$.



11. Return to Symbolic view and delete F2(X) and F3(X).
12. Press **Plot** to return to Plot view. Tap **Menu** and **Sketch**. This time, sketch a line that is parallel to the first line.
13. Record your two linear equations below and discuss what is the same and different about them this time.



Teacher Notes for Introduction to Linear Functions

This activity gives students a more graphical way to approach an understanding of linear equations of the form $y=mx+b$. Rather than focusing on entering linear functions symbolically and then examining their graphs, this activity turns that method on its head and lets students sketch lines with their fingers and then examine the resulting equations.

The goal of the activity is to get students to see that:

1. linear functions are of the form $y=mx+b$, where m and b are real numbers;
2. b is related to the point where the line crosses the y -axis and
3. m is related to the steepness of the line.

Formal mathematical definitions of y -intercept and slope will come later.

Extensions

1. Have students draw lines that go down to the right, up to the right, and straight to the right. Have them record their observations about the possible values for the slope of a non-vertical line.
2. Have students draw lines that go through the origin and make a conjecture about the possible values of the y -intercept b .
3. Have students draw perpendicular lines and examine their slope-values.
4. Challenge students to draw vertical lines and make a conjecture about why this is not possible in the Function app.

Answer Key

5. Both equations have the same y -intercept.

10. Answers will vary, but they will have the same form:

$$Y=m_1X+c$$

$$Y=m_2X+c$$

$$Y=m_3X+c$$

13. Answers will vary, but they will have the same form:

$$Y=mX+c_1$$

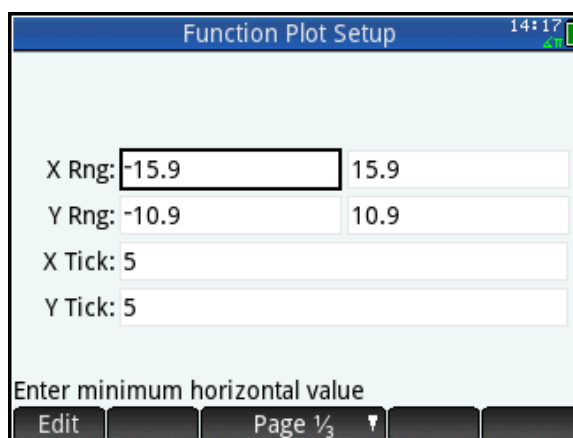
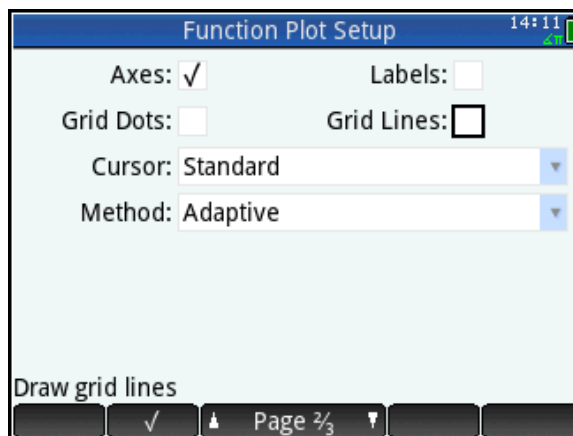
$$Y=mX+c_2$$

The lines have the same slope but different y -intercepts.

A Bridge to Quadratic Functions

In this activity, we start with the image of a bridge as a background. We then scale and move the axes, sketch the arch of the bridge, and study its equation.

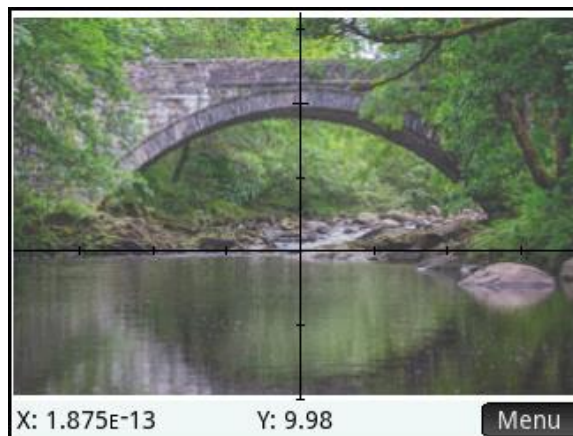
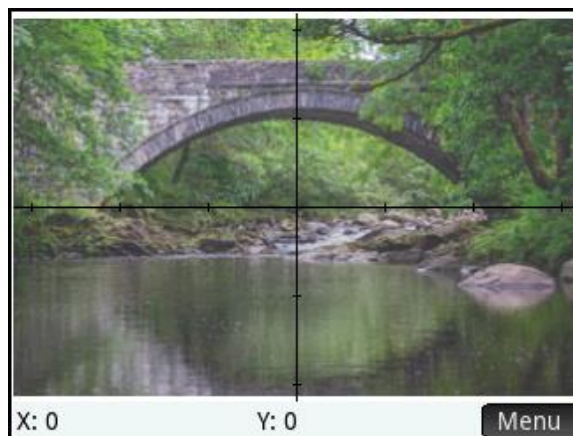
1. Press **Apps Info** to open the Apps Library. With the **Function** app selected, tap **Reset** to reset the app. If you wish, you can save your work first by tapping **Save**, entering a name, and tapping **OK** twice.
2. Press **Shift Plot Setup** to open Plot Setup. Scroll to Page 3 (or tap the page down menu key).
3. On Page 3, tap and select **Best Fit**, then scroll through the list of images until you see **Stream.jpg** as shown to the right.
4. Scroll back to Page 2 and uncheck both **Grid Dots** and **Grid Lines**, as shown in the figure.
5. Finally, scroll back to Page 1 and set both Xtick and Ytick to 5.



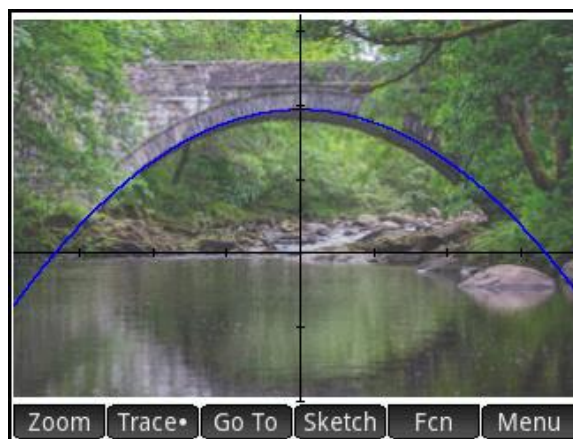
6. Press **Plot Setup** to see Plot view with the bridge image as background. You will see a message that no functions are checked. Tap **OK** to dismiss the message.

It is known that the arch of the bridge is approximately 10 meters above the water at its highest point. We are interested in the width of the arch.

7. Drag with your finger to move the axes where you want them. Pinch diagonally until you get the correct width at the base of the arch. You may have to do this a few times to get the axes and scale right where you want them.
8. Tap on the x-axis at the places where the arch would intersect them and use these values to estimate the width of the arch. Record your answer:



9. Tap **Menu** to open the Plot view menu and tap **Sketch**. Now draw the arch of the bridge with your finger. If your first attempt needs improvement, just draw again with your finger until you get a curve you like. Tap **OK** twice when you are done.
10. Tap **Fcn** and select **Transform...** to view the equation of the arch you drew with your finger. This equation is of the form $Y=A*X^2+C$, where A and C are real numbers. Which one is close to the height of the arch?



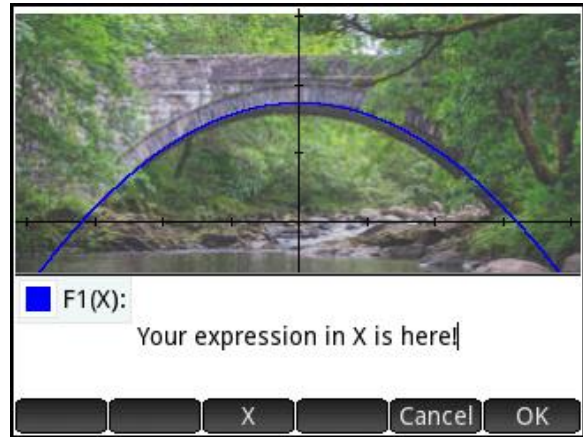
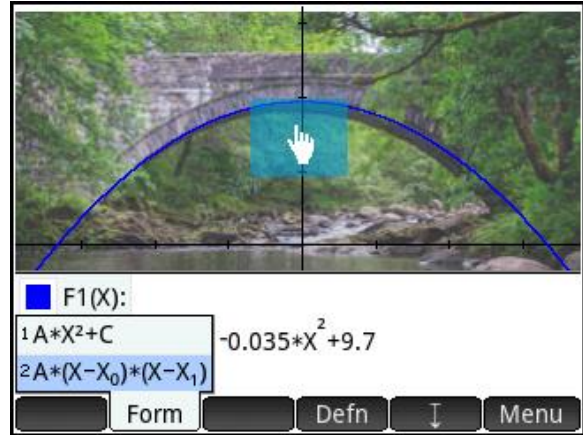
11. Drag to move your graph and pinch horizontally to dilate the graph until it fits the arch as well as you can make it. You can also edit the numbers in the expression directly.
12. Tap **Form** and select **A*(X-X₀)*(X-X₁)**. Record the values of X₀ and X₁. What do you notice about these values?

X₀ = _____

X₁ = _____

What I notice is:

13. In the image, you can see much of the reflection of the arch of the bridge in the water. Return to Symbolic view and make F2(X) the reflection of F1(X). Then go to Plot view to see the reflection plotted.



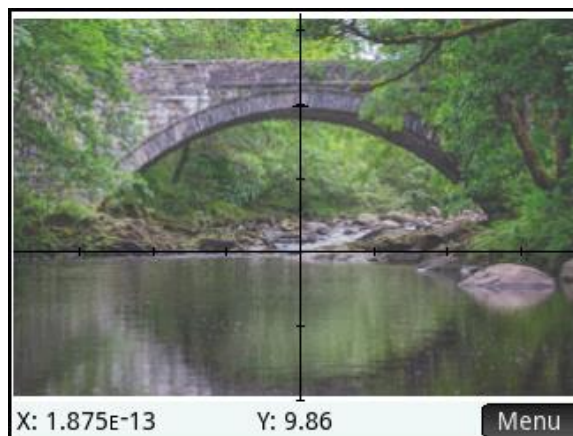
Teacher Notes for A Bridge to Quadratics

This activity motivates an introduction to quadratic functions by relating the parameters in the equation to the shape of the graph. As in the *Introduction to Linear Functions* activity, we start graphically by scaling and moving the axes to reflect the known dimensions of the bridge and then sketching the arch with our finger to produce the expression. After that, we transform the graph to get as close as we can, then change the form of the expression.

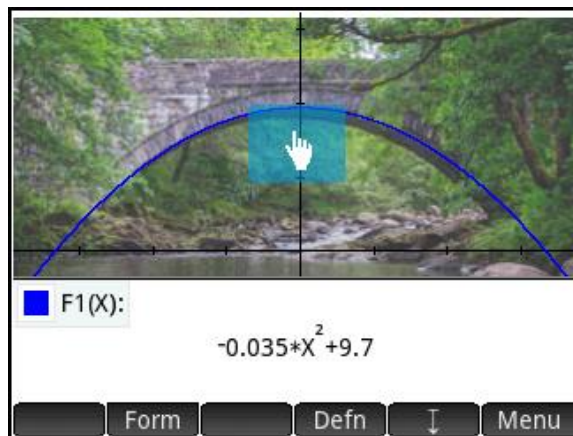
Answer Key

7. In our figure, the arch of the bridge is about 9.9 meters above the surface of the water. The exact numerical value is not the point here.

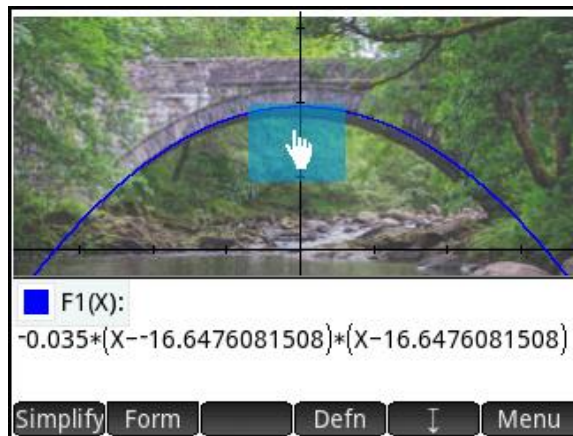
8. The width of the arch is approximately 30m.




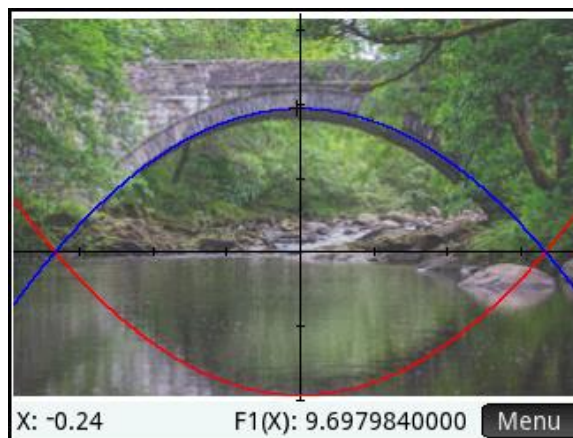
10. Answers will vary and that is OK. We made the graph and its equation off by a bit here to make the point that the graph is an approximation. In our figure, the equation is $Y = -0.035 \cdot X^2 + 9.7$. The height in meters is given by the constant 9.7.



12. Again, answers will vary and that is OK. Here, $X_0 = -16.65$ and $X_1 = 16.65$, so the width of the arch is approximately 33m. The point here is that the difference between the roots estimates the width of the arch.



13. Press  for Symbolic view and define $F2(X) = -F1(X)$. The result is shown in the figure.



Extension

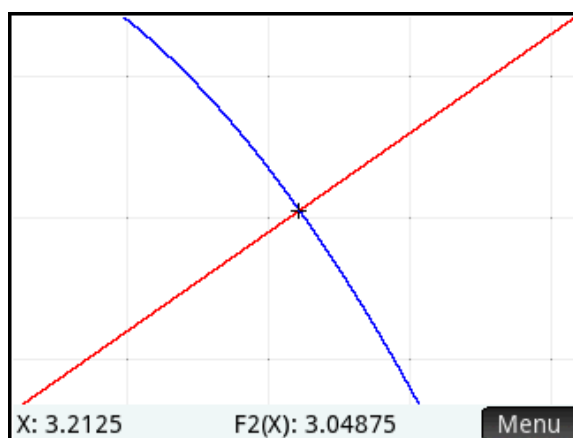
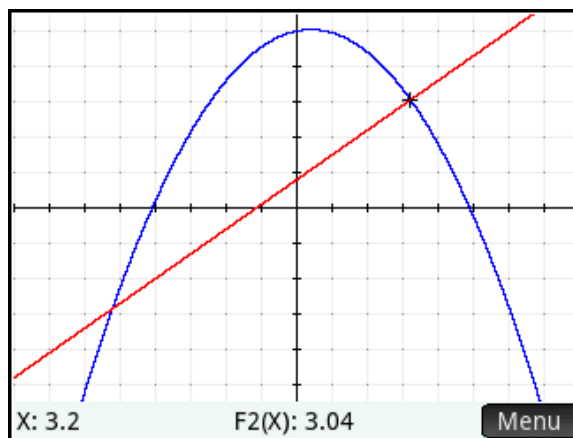
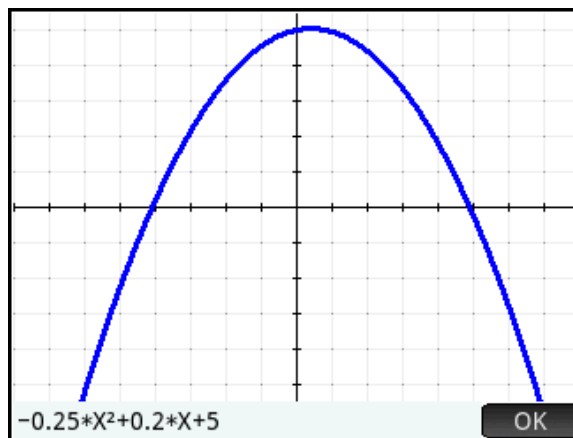
Note that X_0 and X_1 are $\pm \sqrt{\frac{C}{-A}}$. Have students verify the equivalence of AX^2+C and $A(X-X_0)(X-X_1)$ by expressing X_0 and X_1 in terms of C and A .

The Intersection of Two Curves

For our next example, we will draw a parabola and a line and find their intersections in three different ways.


1. Press **Apps** and tap on the **Function** App icon. The app will open in Symbolic view, where you can enter up to ten functions. Press **Plot** to enter Plot view. You will see an error message saying that there are no functions to plot. Tap **OK**, because we are going to draw our own functions.
2. Tap **Menu** to open the Plot view menu, then tap **Sketch**.
3. Draw a parabola with your finger. The HP Prime will show the trail of your finger. If you like the sketch, tap **OK**; otherwise, just draw again with your finger until you get the parabola you want. When you tap **OK**, the function expression is copied to Symbolic view and the graph of the function replaces your sketch.
4. Draw a line that intersects your parabola twice. Tap **OK**.
5. Tap **OK** to exit Sketch mode.
6. Tap at one of the intersections and the tracer will move there. In the figure to the right, there is an intersection in the neighborhood of (3.2, 3.04).
7. Now press **Ans** twice to zoom in on the cursor. In the figure to the right, we have zoomed in on the cursor twice and then moved the cursor closer to the intersection point. Our new estimate of the intersection is (3.2125, 3.04875).

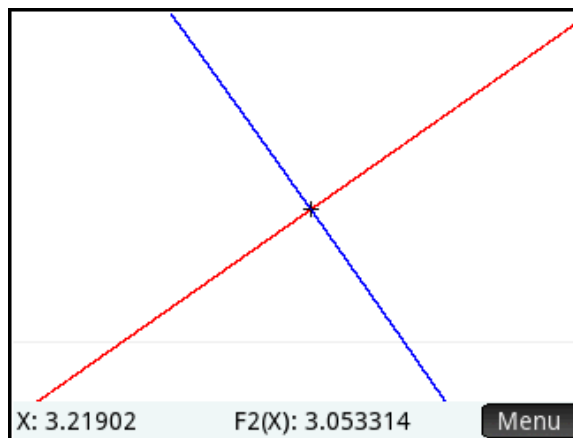
This is a dramatically different technological experience for finding the intersection of two curves- but one that is quite valuable and goes straight to the heart of the process.




8. In the figure to the right, our new approximation for the intersection is (3.219, 3.053) after a few zooms.

You can also zoom in on the intersection using a pinch gesture.

9. Press  to zoom out on the tracer back to the original window.



10. Press  to enter Numeric view. With the cursor in the x-column, enter 3 to see values near x=3. Tap on the x-value 3.2, which is closest to our intersection.


Function Numeric View		
X	F1	F2
2.8	3.6	2.76
2.9	3.4775	2.83
3	3.35	2.9
3.1	3.2175	2.97
3.2	3.08	3.04
3.3	2.9375	3.11
3.4	2.79	3.18
3.5	2.6375	3.25
3.6	2.48	3.32
3.7	2.3175	3.39

3.2

11. Press   for Numeric Setup and change **Num Zoom** to 10.

Function Num Setup	
Num Start:	2.8
Num Step:	0.1
Num Zoom:	10
Num Type:	Automatic

Enter table zoom factor

12. Press  to zoom in on this value.
 After each zoom, tap the x-value that is closest to the intersection before zooming again.





In the figure to the right, we have zoomed in a number of times to estimate our intersection near (3.219, 3.053).

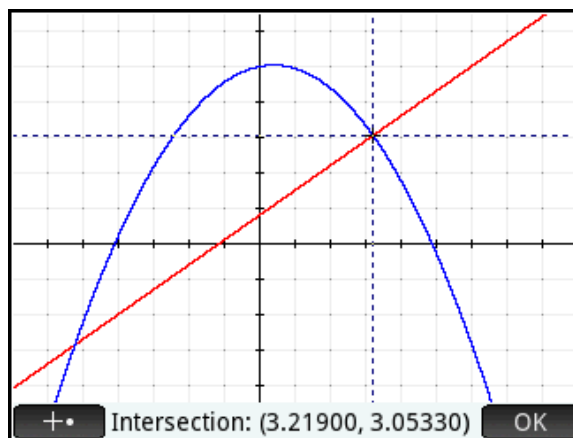
You can also zoom on a row in the table using a pinch gesture.


Function Numeric View		
X	F1	F2
3.214	3.060351	3.0498
3.215	3.05894375	3.0505
3.216	3.057536	3.0512
3.217	3.05612775	3.0519
3.218	3.054719	3.0526
3.219	3.05330975	3.0533
3.22	3.0519	3.054
3.221	3.05048975	3.0547
3.222	3.049079	3.0554
3.223	3.04766775	3.0561




3.219

Zoom More Go To Defn

13. Return to Plot view (press ) , tap  to open the Plot view menu, then tap  and select **Intersection**. You will be prompted to select the other function. Tap  to see the intersection at (3.219, 3.0533).



All the functions available under  in Plot view are available as app functions in Home view.

14. Press  for Home view.
 15. Press  to open the Toolbox menus and select .
 16. Tap **Function** and select **ISECT**.
 17. To find the intersection of F1 and F2 near 3, complete the command as shown. The x-value of the intersection is returned. F1 and F2 of that x-value differ by 1 in the last decimal place.

Function	
ISECT(F1,F2,3)	3.21900462195
F1(3.21900462195)	3.05330323537
F2(3.21900462195)	3.05330323536

Sto ▶

Having a number of diverse but parallel technological experiences all focused on a key concept can help students develop a strong conceptual understanding of mathematics. HP Prime also supports multiple ways of performing common tasks, which allows students to pick the one that feels most natural to them.

Teacher Notes for The Intersection of Two Curves

This activity focuses on giving students multiple technological experiences that get to the heart of finding the intersection of two curves. From the graphical perspective, students are first asked to create two intersecting functions. Then they are asked to zoom in on the curves at the intersection point and move the cursor as close as they can to that point. This process is then repeated to estimate the intersection point as closely as possible. Then, from the numerical perspective, students zoom in on a row in a table of values for the two functions to find the x-value where the two functions have the same value. Again, after the zoom, the student must find and select the row in the table that best represents the intersection point. The process is then repeated until the same estimate is reached as was found in the graphical perspective. After these experiences, the intersection commands are introduced.

This activity is a teacher activity rather than a student one. Its purpose is to get teachers to think about the technological experiences they want their students to have. In this case, having parallel graphical and numerical experiences both underscore the essential elements of estimating the intersection point of two curves. In addition, teachers learn that the commands under **Fcn** are available to them in Home view. Note that the results of these commands are also available in variables that have the same name as the functions.

Extensions

1. Make $F3(X)=F2(X)-F1(X)$ and look for the root of $F3(X)$ both graphically and numerically.
2. After defining $F3(X)$, go to CAS and enter $F3(x)=0$. Then use the collect command to factor the result. One of the factors will be $(x-3.219\dots)$.

The image displays two screenshots from a HP Prime calculator interface.

The left screenshot, titled "Function Symbolic View", shows a list of functions:

- $F1(X) = -0.25 \cdot X^2 + 0.2 \cdot X + 5$
- $F2(X) = 0.7 \cdot X + 0.8$
- $F3(X) = F2(X) - F1(X)$
- $F4(X) =$ (empty)
- $F5(X) =$ (empty)
- $F6(X) =$ (empty)
- $F7(X) =$ (empty)

The right screenshot, titled "CAS", shows the following input and output:

Input: $F3(x)=0$

Output: $0.7 \cdot x + 0.8 + 0.25 \cdot x^2 - 0.2 \cdot x - 5 = 0$

Command: $\text{collect}(0.7 \cdot x + 0.8 + 0.25 \cdot x^2 - 0.2 \cdot x - 5 = 0)$

Result: $0.25 \cdot (x - 3.21900462195) \cdot (x + 5.21900462195) = 0$

An Infinite Series

Suppose we wish to find the sum of the infinite series $1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \frac{9}{16} \dots$

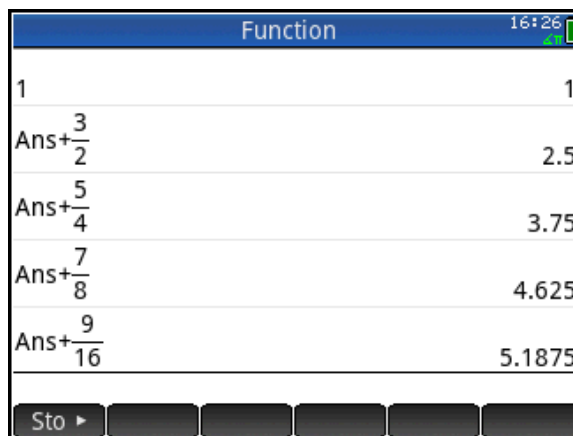
To begin with, we can compute the first few partial sums.

1. Press to enter the Home view.
2. Enter the first term: press .
3. Press .
4. Press to cycle through its decimal, proper fraction, and improper fraction forms

If no entry from Home History is selected, operates on the last answer. You can tap on an entry to select it and then will operate on that selection.

5. Continue as shown to the right until you have the first 10 partial sums. Use to compare the decimal and fractional representations of the partial sums. Enter your list of partial sums in the table below.

N	Sum
1	1/1 or 1
2	5/2 or 2.5
3	
4	
5	
6	
7	
8	
9	
10	



5a. Do you think there is a limit to the sum as N increases? Write your conjecture in the space below.


6. Look carefully at the series again: $1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \frac{9}{16} \dots$


6a. What do you notice about the numerators? Think of 1 as 1/1. Write your response in the space below.


6b. What do you notice about the denominators? Again, write your response in the space below.

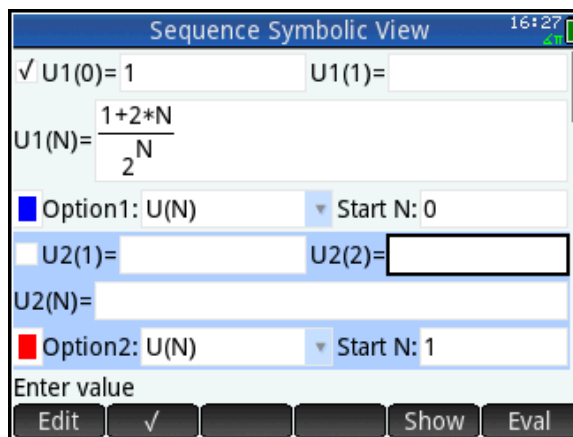
6c. Let A(N) be the function that generates each term in the series for N=0, 1, 2, 3,... Write the formula in N that defines A(N).

A(N) = _____

7. Press  to open the Apps Library. Scroll down and start the Sequence App.
8. The app opens in Symbolic view, where you define up to 10 sequences: U1-U9 and U0.
9. Change **Start N** to 0 instead of 1. Then let U1(0)=1.
10. Finally, enter your formula for the Nth term in U1(N)

Hint: press  and choose the fraction template to start.

11. Press  to see a table of values for the sequence. Verify that the first few entries agree with the fractions 1/1, 3/2, 5/4, 7/8, and 9/16 given at the start of this activity for our sequence.

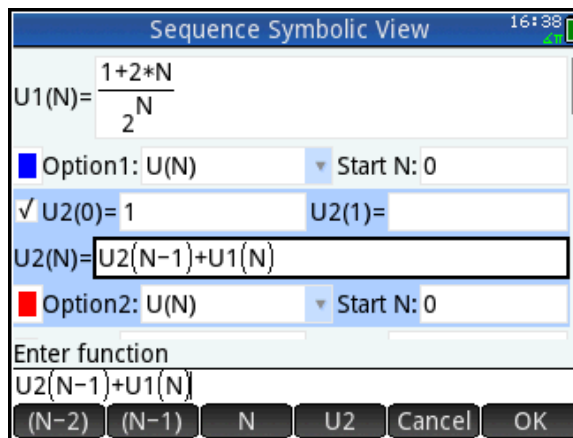


N	U1
0	1
1	1.5
2	1.25
3	0.875
4	0.5625
5	0.34375
6	0.203125
7	0.1171875
8	0.06640625
9	0.037109375
0	

We will now create a second sequence to display the partial sums of the first sequence.

12. Press **Symb** to return to Symbolic view. In $U_2(N)$, let Start N be 0, as in $U_1(N)$.
13. $U_2(1)=1$ is the first partial sum.
14. $U_2(N)=U_2(N-1)+U_1(N)$. This says each partial sum is the previous partial sum plus the nth value of the U_1 sequence.
15. Press **Num** to see the table of values. Verify that the first 10 partial sums agree with your table from #5.
16. Scroll down through the table of values. What do you think is the limit of these partial sums? In other words what is the sum of our infinite series?

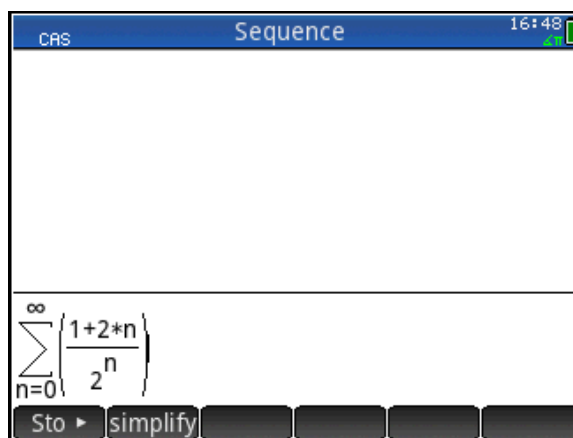
The limit is _____.



N	U1	U2
0	1	1
1	1.5	2.5
2	1.25	3.75
3	0.875	4.625
4	0.5625	5.1875
5	0.34375	5.53125
6	0.203125	5.734375
7	0.1171875	5.8515625
8	0.06640625	5.91796875
9	0.037109375	5.955078125
0		

What is the limit as n approaches ∞ ? Use the CAS to find $1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \frac{9}{16} \dots = \sum_{n=0}^{\infty} \frac{1+2n}{2^n}$

17. Press **CAS** to open the CAS view. Press **Units** to open the Template menu and select the summation template
18. To enter $n=0$, press **ALPHA**, **()**, **0**
19. To enter ∞ at the top, tap on the top square in the template, press **Shift**, **9** and tap on ∞ .
20. Tap on the right template box and enter the rational expression. Hint: use the fraction from the Template menu.
21. When you are done, press **Enter** and record the value returned.



The sum is: _____

Teacher Notes for An Infinite Series

This short activity starts by exploring the partial sums of an infinite series in Home view, then introduces the Sequence App to automate the process. Once a conjecture on the limit of the partial sums is reached, the CAS is invoked to verify the conjecture.

Answer Key

5.

N	Sum
1	1
2	5/2 or 2.5
3	37/8 or 4.625
4	83/16 or 5.1875
5	177/32 or 5.53125
6	367/64 or 5.734375
7	749/128 or 5.8515625
8	1515/256 or 5.91796875
9	3049/512 or 5.955078125
10	6119/1024 or 5.9755859375

5a. The limit seems to be 6.

6a. The numerators are all odd numbers.

6b. The denominators are all powers of 2.

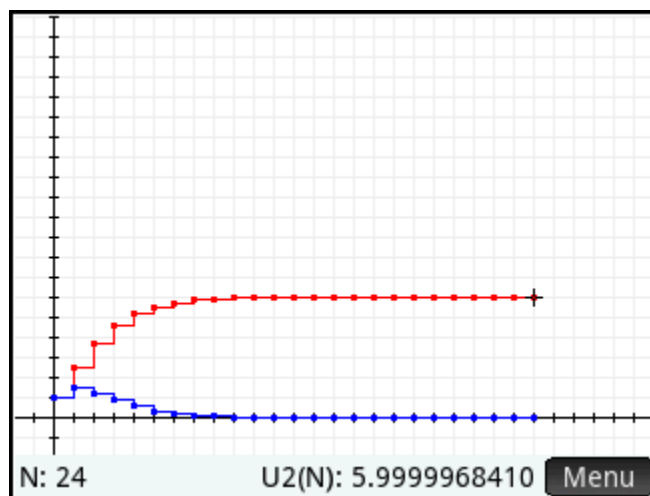
6c. $A(N) = (1 + 2^N) / (2^N)$ for $N = 0, 1, 2, \dots$

15. The limit is 6

$$20. \sum_{n=0}^{\infty} \frac{1+2n}{2^n} = 6$$


Extensions

- Use Plot view to give students a visual representation of both sequences. The figure shows the first 24 terms.




Solving Trigonometric Equations

How many solutions are there to the equation $\sin(x) = \frac{\sqrt{3}}{2}$ and what are they? This activity will help you answer those questions for yourself.

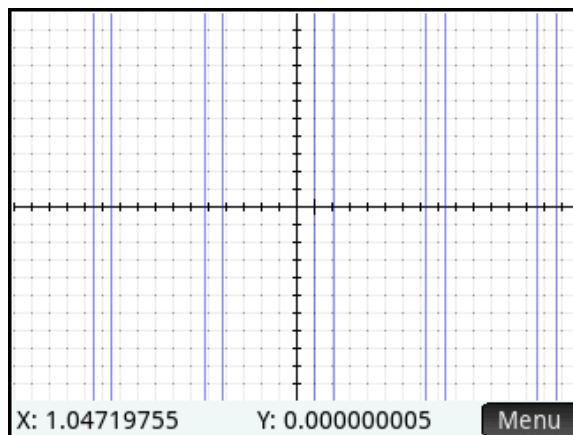
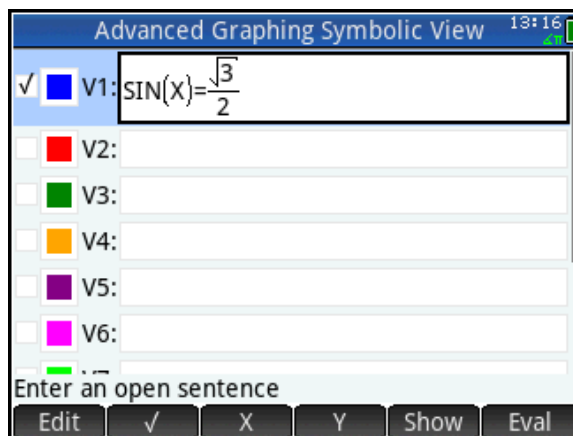
1. Press  and tap on the Advanced Graphing app icon; the app opens in Symbolic view.

Here, you can enter up to 10 equations or inequalities.

2. In V1, enter $\sin(x) = \frac{\sqrt{3}}{2}$

3. Press  to see the graph.

4. The graph is a set of vertical lines. Record your observations about these vertical lines. Specifically, how many are there and is there a pattern to these lines? You can pinch to zoom in and out as well as drag to explore the set of lines.



The vertical lines suggest solutions of the form $x=c$, where c is a real number. We will now train the tracer to jump from one x -intercept to the next.

The screenshot shows the HP Prime calculator's graphing interface. A graph is displayed with several vertical blue lines. A menu is open, showing options for 'Tracing' and 'Points of Interest'. The 'Points of Interest' menu is currently selected, and the '4 PoI' option is highlighted. Below the graph, the X-axis value is 2.0943951 and the Y-axis value is 0. At the bottom, there is a table titled 'Advanced Graphing Numeric View' showing a list of x-values.

Advanced Graphing Numeric View	
V1	
1.04719755	
2.0943951	
7.33038286	
8.37758041	
13.6135682	
14.6607657	
1.047197551	

8. We can express the entire sequence of solutions with a single expression of the form $x=c + d^n$,


where c is the first term of the sequence, d is the common difference, and n is any integer. Write this expression with the values of c and d expressed in terms of π .

9. What about the remaining x-intercepts: 2.0943, 8.3775, 14.6607, ...? Do these values form an arithmetic sequence? If so, what is their common difference in terms of π ?

10. Express the second set of solutions in a single expression of the form $x=c+d^n$.

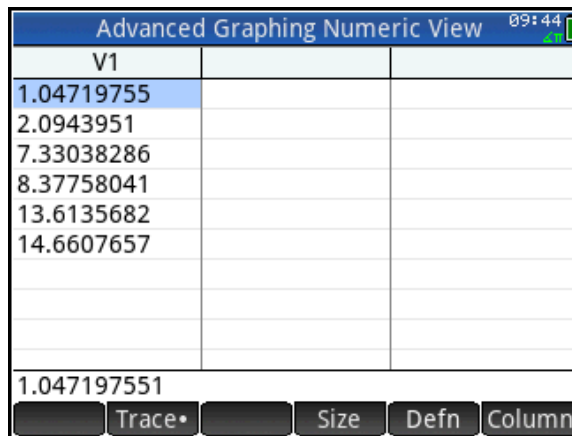
11. Now express the total set of solutions by combining your two expressions in an “or” statement.

12. Why did the equation $\sin(x) = \frac{\sqrt{3}}{2}$ show

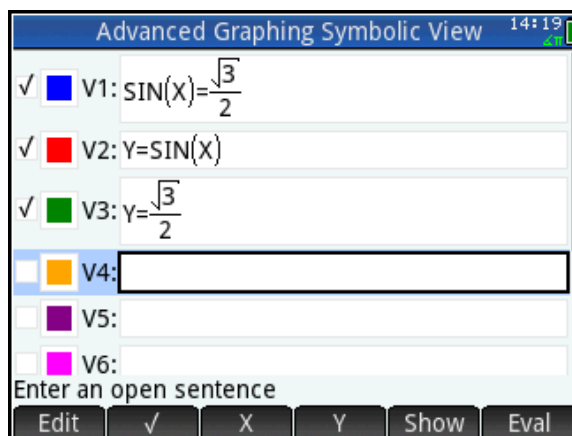
us these particular vertical lines as solutions? Press  to return to Symbolic view and add $Y=\sin(X)$ and

$Y = \frac{\sqrt{3}}{2}$ in V2 and V3. Then press 

to see the bigger picture.



Advanced Graphing Numeric View 09:44		
V1		
1.04719755		
2.0943951		
7.33038286		
8.37758041		
13.6135682		
14.6607657		
1.047197551		
Trace • Size Defn Column		

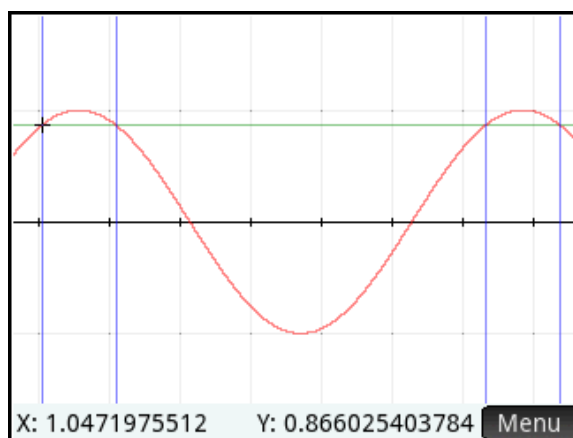


Advanced Graphing Symbolic View 14:19

- ✓ V1: $\sin(X) = \frac{\sqrt{3}}{2}$
- ✓ V2: $Y = \sin(X)$
- ✓ V3: $Y = \frac{\sqrt{3}}{2}$
- V4:
- V5:
- V6:

Enter an open sentence

Edit ✓ X Y Show Eval



Teacher Notes for Solving Trigonometric Equations

In this activity, we look at introducing students to the topic of solving trigonometric equations.

As an example, we take the equation $\sin(x) = \frac{\sqrt{3}}{2}$. If we use this equation to introduce students

to the notion of solving trigonometric equations, what do we want them to notice from the very outset? First and foremost, there are three important aspects to the solution set:

1. There are an infinite number of solutions
2. These solutions are composed of two branches
3. There is a conventional way of expressing these two branches of infinite solutions in a succinct way

This activity focuses students on these three key areas. The rules for quickly expressing solutions to equations involving sine and cosine can be developed later.

Answer Key

4. The graph appears to contain an infinite number of vertical lines. The lines are arranged in pairs, and each pair appears to be the same distance apart. So far, this graphical representation has done a good job of fulfilling our first two requirements above.

5. The equation of the first line is $X = \pi/3$.

6. The equation of the second line is $X = 2\pi/3$.


7. Yes; the common difference is 6.28... or 2π .

10. $X = 2\pi/3 + 2\pi n$


11. The solution is $X = 2\pi/3 + 2\pi n$

A Special System of Linear Equations

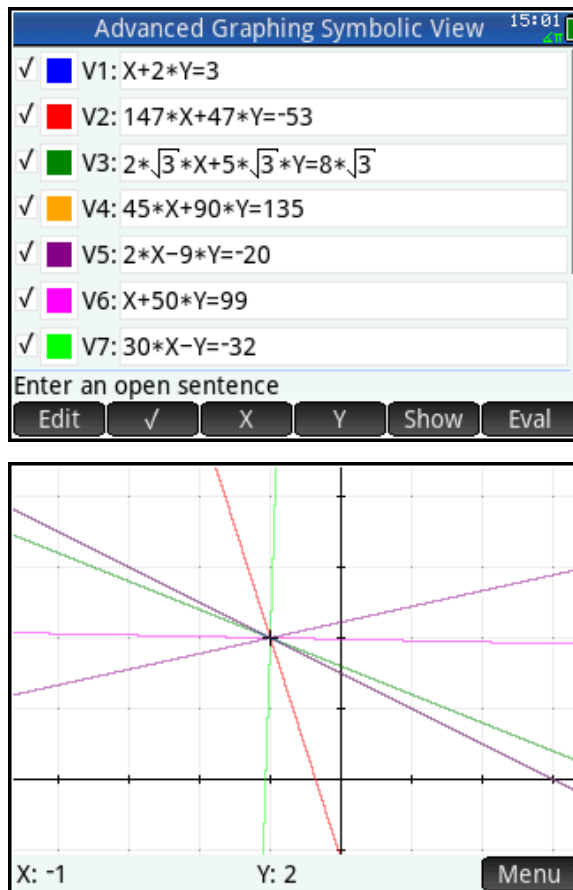
Suppose you have a set of linear equations in Standard Form: $Ax + By = C$. Suppose further that the coefficients of each equation (A , B , and C) form an arithmetic sequence. What surprising property do they all have in common?

1. Press  to open the App Library and select the **Advanced Graphing** app.
2. Create several linear equations, each of whose coefficients form an arithmetic sequence. The figure to the right shows a varied set of 7 such equations.

I like to challenge students to come up with their own outlandish examples and use them!

3. Press  to see the graphs.

It is surprising the first time one sees that these lines are all concurrent at the point whose coordinates are $(-1, 2)$! Students will be curious about the mathematics that shows why this must be true!



How do we re-write the equation $AX+BY=C$ so that it reflects the fact that A , B , and C form an arithmetic sequence? Start with AX . Then B must be the sum of A and some value. Let's write $AX + (A+B)Y$. So the constant C must be B greater than this, or $A+2B$. So now we have $AX + (A+B)Y = A+2B$.

By inspection, $Y=2$ in order to have $2B$ on both sides. That leaves us:

$$AX+2(A+B)=A+2B \text{ or}$$

$$AX + 2A + 2B = A + 2B$$

Now we have $2A$ on the left but only $1A$ on the right. So $X= -1$ is the only way to make the equation true. Thus any equation with this property contains the point $(-1, 2)$! Amazing!

Teacher Notes for A Special System of Linear Equations

The mathematical problem in this activity was featured at an NCTM National Conference lesson study presentation by Dan Kennedy. Since the Advanced Graphing App can graph linear equations in standard form, it is perfect for the activity. The fact that a line whose parameters in standard form are consecutive terms in an arithmetic sequence must contain the point $(-1, 2)$ is surprising!

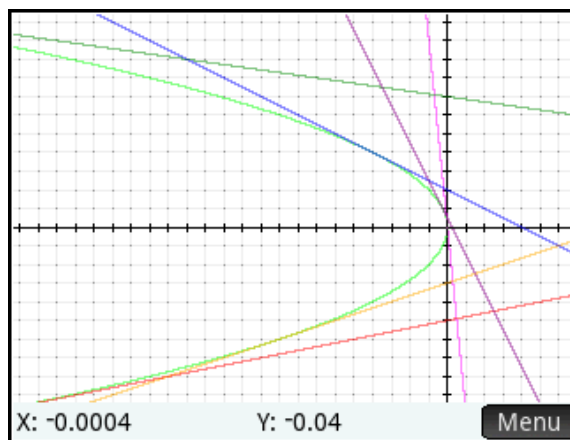
Although this activity was written as a teacher activity, it follows a key recipe for creating a good, interesting student activity in mathematics education:

1. The activity states a problem that is simple enough for all students to understand
2. There is a surprising result
3. The only way to explain the surprising result is through sound mathematical reasoning, proof, and discourse

Extensions



1. What happens if the parameters are consecutive terms of a geometric sequence instead?

In this case, the lines are all of the form $AX+BY=AB^2$, where $B \neq 0$. Each line of this form will have a y-intercept at B and a root at B^2 . Think of an infinite number of pairs of lines. The lines in each pair have a common root at B^2 and y-intercepts at $\pm B$. These lines form a pencil of lines that all seem to be tangent to the parabola $x = -\frac{1}{4}y^2$. This conjecture is left to the reader to prove.




Identities, Equivalence, and Conditions

Students usually just try to memorize basic identities. But HP Prime has an app that lets you test mathematical statements in 1 or 2 variables!

1. Press  to open the App Library and select the Advanced Graphing app. The app opens in Symbolic view, where you can enter up to ten mathematical statements. These can be equations or inequalities.
2. In V1, enter $\sqrt{X^2} = X$. Tap  for the equal sign.

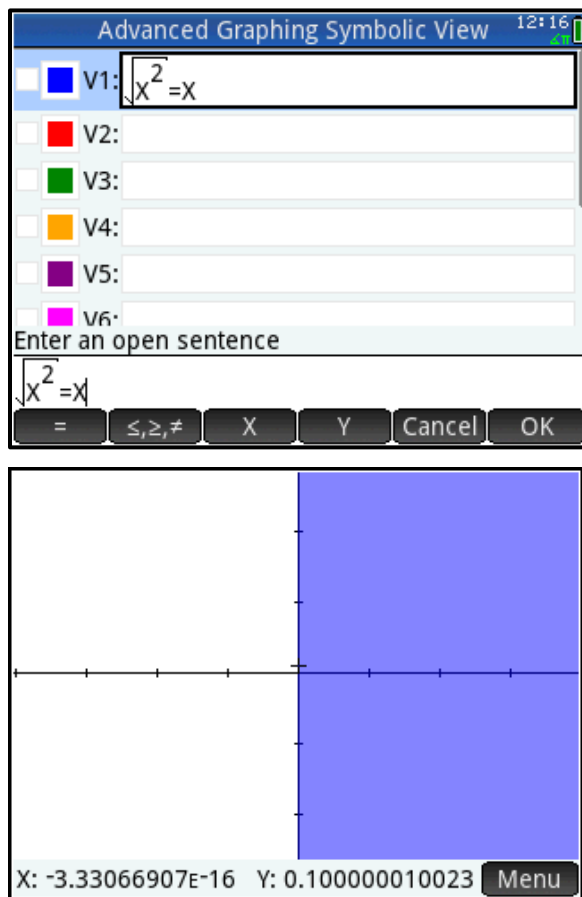
What do you expect to see for the graph of this equation?

3. Press  to see the graph. The statement was an equation, but the graph looks like the graph of an inequality, specifically $X \geq 0$.

Students must realize that the graph means $\sqrt{X^2} = X$ iff $X \geq 0$; once they see this, they have a relatively easy way of testing most of the identities they will meet in high school mathematics.



4. Try the following:

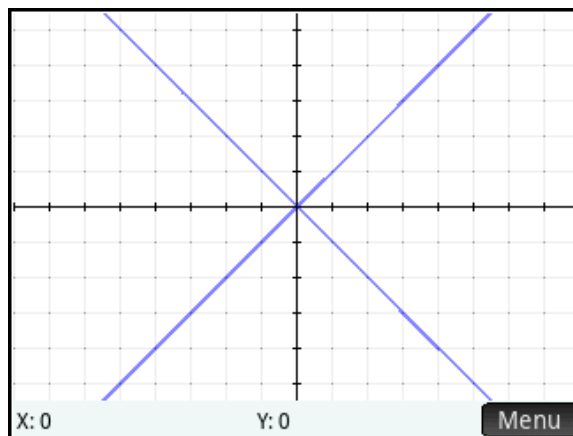
- $\sqrt{X^2} = |X|$
- $\sin^2(x) + \cos^2(x) = 1$
- $\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$



Statements in Two Variables

If $X^2 = Y^2$ does it follow that $X=Y$? That is, are $X^2 = Y^2$ and $X=Y$ equivalent expressions?

5. Press  and select the Advanced Graphing app
6. Enter $X^2 = Y^2$ in V1
7. Press . The graph is shown to the right.



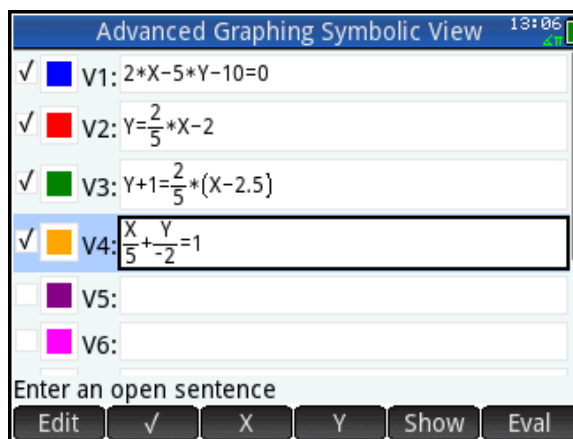
It seems there are 2 possibilities, and only one of them is that $X=Y$. Students must identify the other branch as $X= -Y$.


If $X^2 = Y^2$ then either $X=Y$ or $X= -Y$.

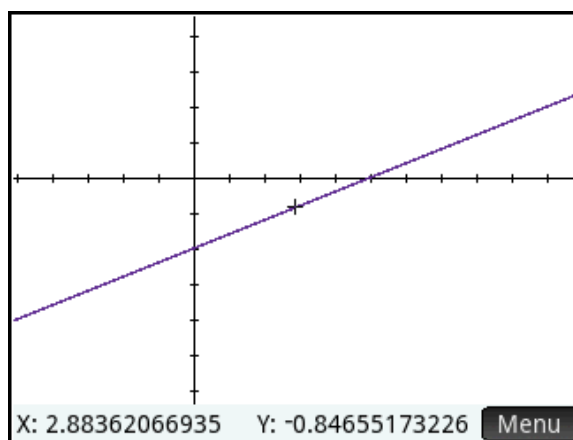
8. Try the following:
 - If $X^3=Y^3$, does it follow that $X=Y$?
 - $\sqrt{x^2 + y^2} = x + y$
 - $|X| + |Y| = |X+Y|$
 - $|X+Y|=X+Y$

Graph it Your Way!

In the figure to the right, a single line has been represented in 4 symbolic forms students usually encounter in high school. With the Advanced Graphing app, all of these forms are considered valid.



Press  to see the graph of just the one line.



This unparalleled level of flexibility in the Advanced Graphing App allows students to test an identity or the equivalence of two different expressions. It also allows students to solve problems their own way, without having to conform to structures imposed by machines.

Teacher Notes for Identities, Equivalence and Conditions

This is another teacher activity rather than a student activity. Most activities involving graphing calculators focus on functions. With the Advanced Graphing app, it is important to note that we can now study equivalence in a more general sense.

In this activity, participants are exposed to graphical representations of two notions:

1. Equivalence: the notion that two different expressions are equal for all real numbers
2. Conditional equivalence: the notion that two different expressions are equal for some subset of real numbers

While the graphs associated with such statements are not proofs of equivalence or conditional equivalence, they do offer immediate visual insights to students who can then make and prove their own conjectures.

Answer Key

2-3. In the Cartesian plane, the graph of $\sqrt{X^2} = X$ is the half-plane where $X \geq 0$. Teachers and students who are used to graphing only functions may not have a clear expectation for the graph of an expression that is only in X . The fact that the graph of an equation appears to be the graph of an inequality leads directly to the notion of conditional equivalence, that

$\sqrt{X^2} = X$ if and only if $X \geq 0$.

4. These are all identities

- $\sqrt{X^2} = |X|$
- $\sin^2(x) + \cos^2(x) = 1$
- $\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$;



7. This is an example of conditional equivalence in two variables. $X^2 = Y^2$ if $X=Y$ OR if $X=-Y$.

8.



- If $X^3=Y^3$, does it follow that $X=Y$? Yes, this is an identity
- $\sqrt{x^2 + y^2} = x + y$ For this one, you may want to turn off the axes. The statement is true if either $x=0$ and $y \geq 0$ or if $y=0$ and $x \geq 0$.
- $|X| + |Y| = |X+Y|$ is true for X and Y in the first and third quadrants. That is, the statement is only true if both X and Y have the same sign.
- $|X+Y|=X+Y$; this one is true if and only if $Y \geq -X$. The graph is the upper half-plane bounded by $Y=-X$.

Operating on Functions


In this activity, we look at two linear functions and estimate their sum and their difference.

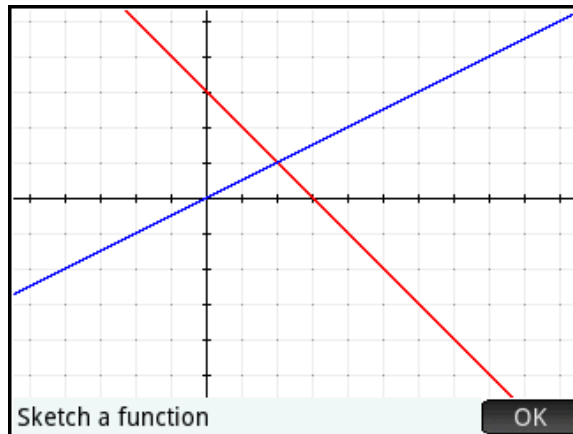
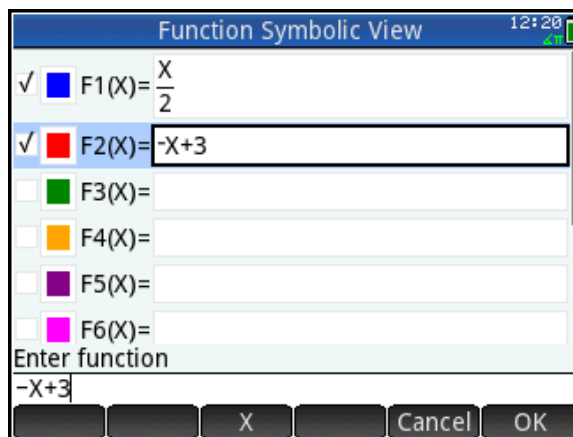
1. Press  to open the Apps Library and start the Function App. Let $F1(X)=X/2$ and $F2(X)=-X+3$, as shown on the figure.
2. Press  to see the graphs of these two linear functions.

We wish to draw the sum of these two linear functions.

3. Tap  to open the Plot view menu and tap .

Before you draw, think carefully about the sum of these two functions. For each x -value, look at the corresponding y -values and add them up. For what x -values will this be easiest? How many x -values do you need before you have enough to draw the graph?



4. Draw the sum with your finger. Then tap  twice. The equation for your sum is now in $F3(X)$.
5. Answer the questions and record the equation of your sum in the space below.

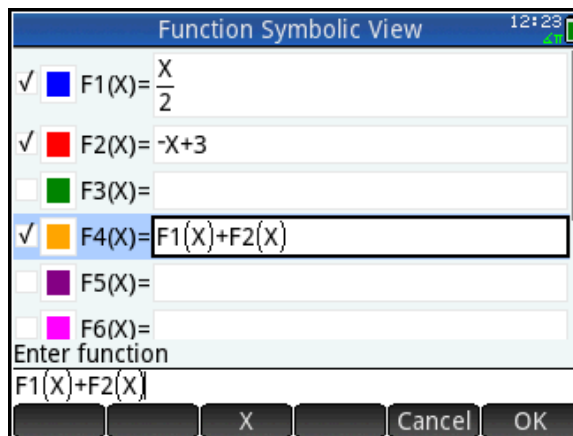


5.1 What x -values will be easiest for summing the y -values?

5.2 How many points do you need before you can draw the sum?

5.3 What is the equation of your sum? $F3(X) =$ _____





6. Return to Symbolic view (press ). Your sum is in F3(X). Let $F4(X)=F1(X)+F2(X)$. Press  to see the sum plotted against your estimate. Were you close?

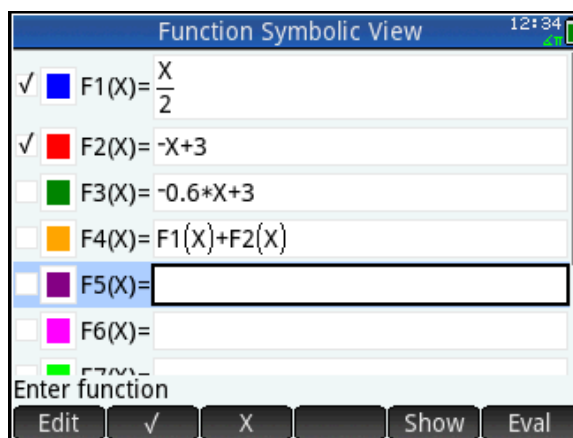


7. What is the equation of the sum?


$F1(X)+F2(X)=$ _____

Now let's try the difference of the two functions, specifically $F1(X)-F2(X)$.

8. Press  for Symbolic view and uncheck F3(X) and F4(X) so they won't be plotted.
 9. Press  for Plot view and tap  to open the Plot view menu and tap .



Before you draw, again think carefully about the difference of these two functions. In this case it's the blue graph minus the red graph. For each x-value, look at the corresponding y-values and subtract them in that order. For what x-values will this be easiest? How many x-values do you need before you have enough to draw the graph?



10. Draw the sum with your finger. Then tap  twice. The equation for your sum is now in F5(X).
 11. Answer the questions and record the equation of your difference in the space to the right.

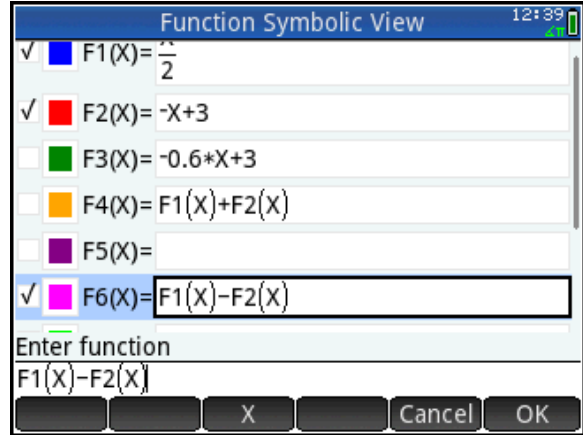
11.1 What x-values will be easiest for subtracting the y-values?

11.2 How many points do you need before you can draw the difference?

11.3 What is the equation of your difference?

$F5(X)=$ _____

12. Return to Symbolic view (press ).
- Your difference is in F5(X). Let $F6(X)=F1(X)-F2(X)$. Press  to see the sum plotted against your estimate. Were you close?



-
13. What is the equation of the difference?

$F1(X)-F2(X)=$ _____

14. From what you have learned, can you write the equation of $F2(X)-F1(X)$ without sketching?

$F2(X)-F1(X)=$ _____

Teacher Notes for Operating on Functions

This activity came from Eduardo Basurto and Eduardo Mancera of SIRVE in Mexico. It is a great way of elevating student thinking from operations on numbers to operations on functions. The graphical approach focuses student attention on the operations instead of the symbols. Here we focus on the sum and difference of two linear functions, but the extensions to the product and quotient are musts as well!

Answer Key

5.1 The two points where one of the lines has an x-intercept are the easiest to visualize the sum (as one of the members of the sum is zero).

5.2 From the symbolic form $AX+B$, students should see that the sum will still be of the same form. Another approach would be to have students write the coordinates of 3 or 4 points on the sum. From there, they should be able to see that those points are collinear.

5.3 $F_3(X) = -X/2+3$ is the sum; answers may vary a bit, depending on how skilled students are at drawing lines with their fingers.

6. Hopefully, the student lines will be close to the sum defined in $F_4(X)$. If not, use the opportunity to discuss what went wrong.

7. The sum is $F_1(X)+F_2(X) = -X/2+3$

11.1 The easiest point to plot is where the lines intersect, as there the difference is zero. By the way, this is a very important idea for students to realize as well and will be important when they study intersections in general. After that, the two roots are the easiest.

11.2 Again, students may see that the difference is still linear in form (with the possibility of zero slope), or they may need to write down the coordinates of 3-4 points before they see their collinearity.

11.3 Their difference should be close to $F_5(X)=1.5X-3$.

12. Hopefully, their sketches were close to the equation above. Again, if they are not, it is an opportunity to discuss what went wrong.

13. $F_1(X)-F_2(X)=1.5X-3$

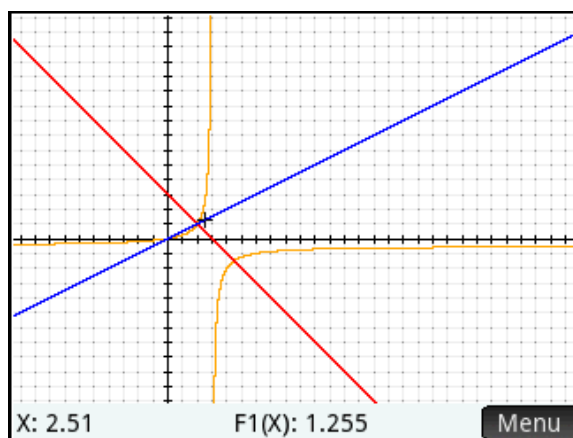
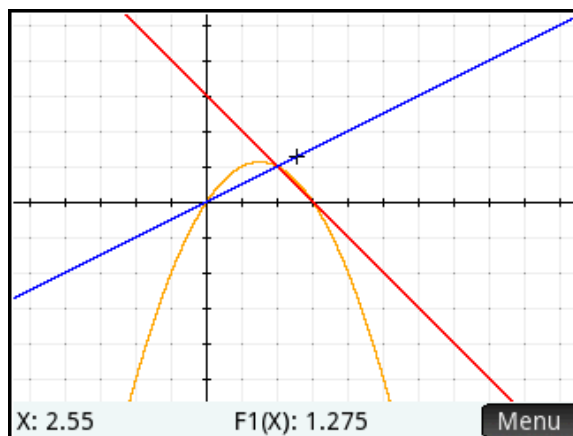
14. $F_2(X)-F_1(X) = -1.5X+3$. You may have to encourage students to look at the equations for $F_1(X)$ and $F_2(X)$ and compare these to the equations for $F_1(X)+F_2(X)$ and $F_1(X)-F_2(X)$ before they come up with $F_2(X)-F_1(X)$ without sketching.

Extensions

1. Have students sketch $F1(X)*F2(X)$. Here the roots become very important, as the product has both roots. This is a great way of recognizing the multiplicative property of zero in context. Then have student look between the roots and establish the sign of each function and the sign of the product. Finally, have student look on either side of the roots, establish the sign of each function and the sign of the product. Their sketches may not match the true graph as well as the sum or difference, but this is instructive as well.

2. $F1(X)/F2(X)$ is trickier, but can lead to some great observations as well:
 - a. The quotient approaches $-1/2$ as x approaches $\pm\infty$, which can be seen by zooming.
 - b. The value of the quotient where the lines intersect is one.
 - c. As x approaches 3 from either side, the denominator goes to zero and makes the quotient extremely large in absolute value.

All of these are great mathematical insights!

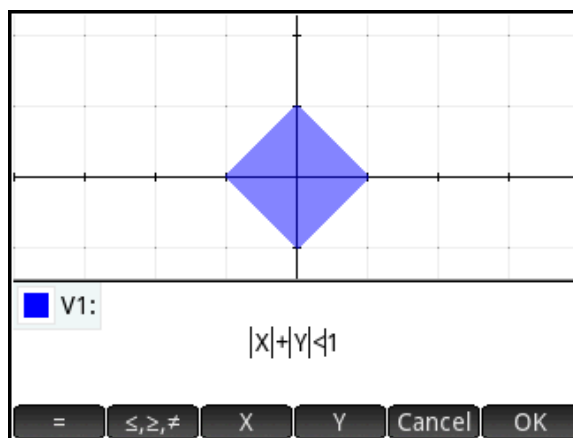
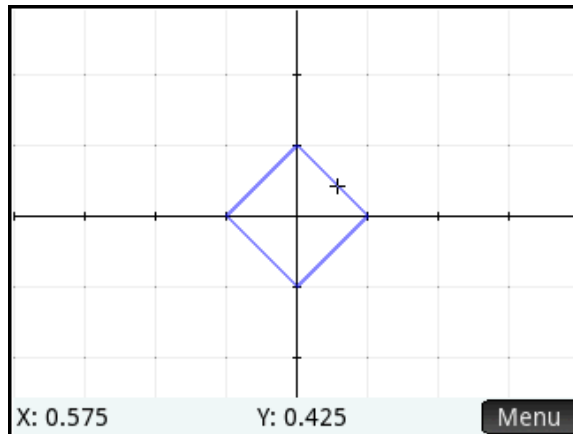
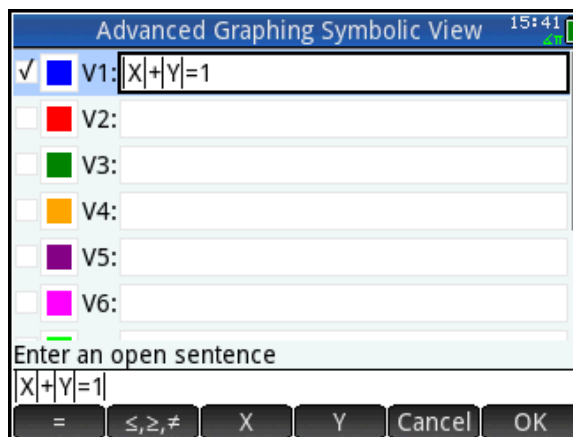


Mathematical Art

In this activity, we encounter some tips and tricks for using the Advanced Graphing App.


Start with a basic shape

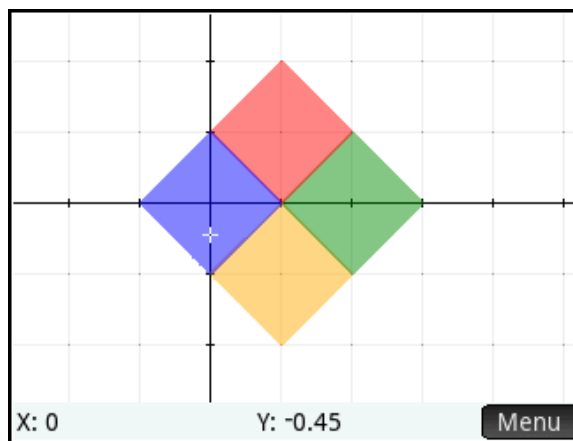
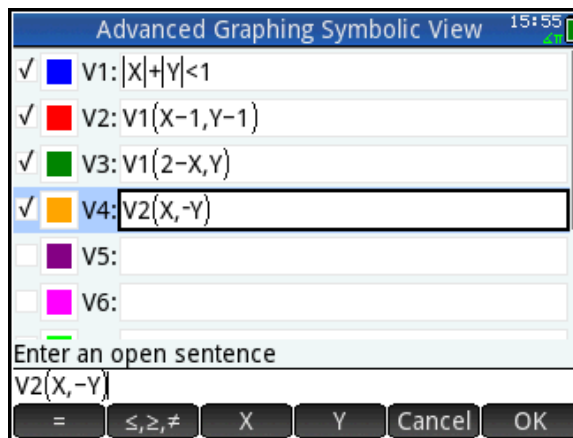
1. Press **Apps** to open the Apps Library. Reset and then start the Advanced Graphing App. In V1, enter $|X| + |Y| = 1$. What do you expect the graph to look like?
2. The graph is a diamond (square)!
3. Tap **Menu** and then tap **Defn**. Tap inside the definition and change the = to <. Hint: tap **\leq, \geq, \neq** to see the less than symbol. Tap **OK** to see the diamond filled-in.



Use geometric transformations!

We can use V1 as a basic shape to create all sorts of designs. In the figure to the right, V2 has been defined to be the translation of V1 up 1 unit and to the right 1 unit, V3 has been defined as the reflection of V1 in the vertical line X=1, and V4 has been defined as the reflection of V2 over the x-axis.


4. Define V2, V3, and V4 as shown.
5. Press  to see the graph.

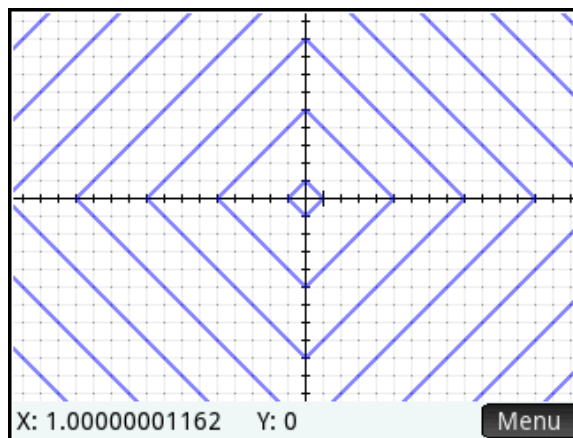


My Favorite Trick!

6. Return to Symbolic view and delete V2, V3, and V4. Now define V1 as

$$\text{SIN}\left(\frac{\pi}{2}(|X|+|Y|)\right) = 1.$$

Press  to see the graph. Now we have concentric diamonds! Notice the right-most vertices are at (1, 0), (5, 0), (9, 0), (13, 0),... Why?



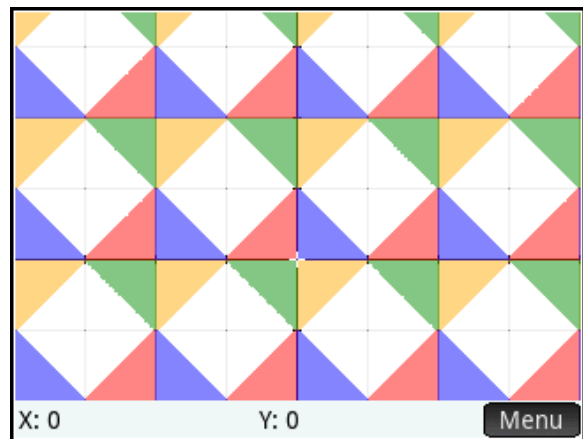
This technique can be expanded to any expression in X and Y: $SIN\left(\frac{\pi}{2}(F(X,Y))\right) = 1$

7. Try $SIN\left(\frac{\pi}{2}(X^2 + Y^2)\right) = 1$. Describe the graph.

8. Make $F(X, Y) = \left(X^2 + \left(Y - \sqrt[3]{X^2}\right)^2\right)$. Can you describe the graph?

Another Trick!

9. Return to Symbolic view and change V1 to $|X \text{ MOD } 2| + |Y \text{ MOD } 2| < 1$. Can you figure out how to define V2, V3, and V4 to get the graph shown?



V2: _____

V3: _____

V4: _____

Teacher Notes for *Mathematical Art*

This activity focuses on techniques for creating graphic designs using the Advanced Graphing App. The first technique is to create a basic shape and then use standard geometric transformation techniques. If V1 contains the definition of the basic shape, then you can enter any of the following (or combinations of them) into V2-V9:

- V1(Y, X) plots the inverse of V1
- V1(2A-X, Y) plots the reflection of V1 over the vertical line X=A
- V1(X, 2B-Y) plots the reflection of V1 over the horizontal line Y=B
- V1(X-A, Y-B) translates V1 A units horizontally and B units vertically
- V1(AX, BY) dilates V1 horizontally by a scale factor of A and vertically by a scale factor of B

NOTE: Although this notation may not be standard, it is easily understood how to use it to create the design you want.

6. $\sin(x)$ has a value of 1 at $x = \pi/2, 5\pi/2, 9\pi/2, \dots$ So $\text{SIN}\left(\frac{\pi}{2}(|X| + |Y|)\right) = 1$ is true when

$|X| + |Y| = 1, 5, 9, 13, \dots$ But $|X| + |Y| = 1$ is a diamond; thus, so are $|X| + |Y| = 5, |X| + |Y| = 9$, etc. So we have a set of concentric diamonds!

7. Similarly, $\text{SIN}\left(\frac{\pi}{2}(X^2 + Y^2)\right) = 1$

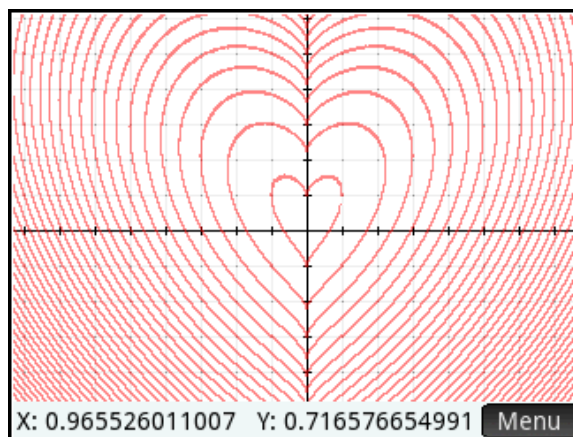
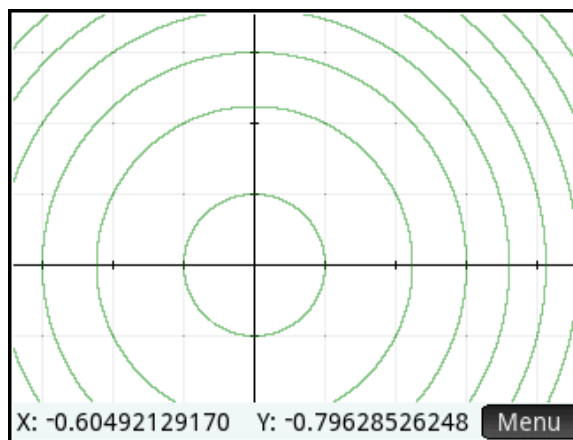
generates a set of concentric circles. In this case, the radii of the circles are

$1, \sqrt{5}, 3, \sqrt{13}, \dots$ So the pattern of 1, 5, 9, 13, ..., $4N+1, \dots$ (where N is a positive integer) persists here as well.

8. This one comes to us courtesy of Mark Howell. It is a set of concentric hearts!

9. Below are the definitions to achieve the pattern of triangles. Using the MOD statement restricts the original diamond to the first quadrant triangular quarter but also repeats that triangle every 2 units vertically and horizontally.


- V1: $|X \text{ MOD } 2| + |Y \text{ MOD } 2| < 1$ (given)
- V2: V1(2-X, Y)
- V3: V2(X, 2-Y)
- V4: V3(2-X, Y)









Peculiar Parabolas

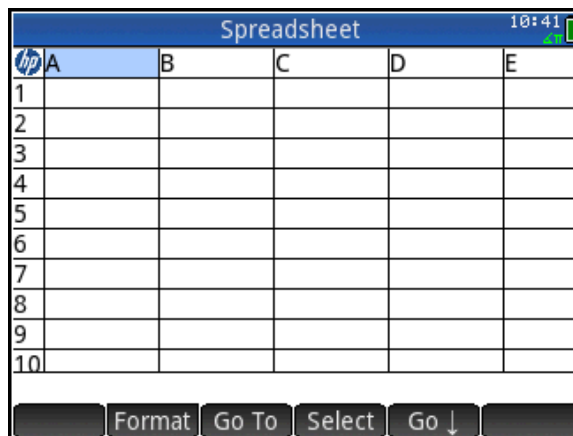
In this activity, we try to construct a parabola with the property that none of the points on the parabola have integers for both coordinates. Do you think this is possible?

To begin with, we know the parabola must have either an x^2 or y^2 term.

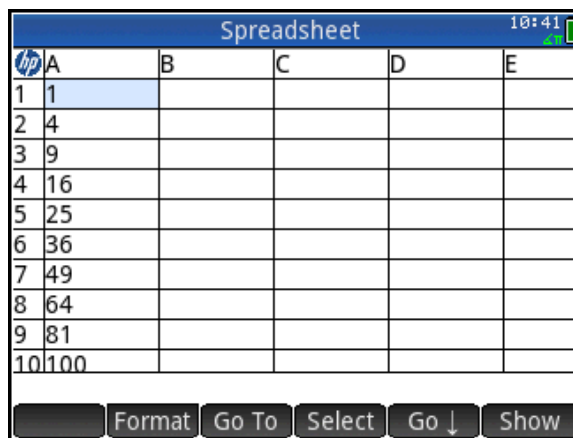
1. Press  to open the Apps Library and start the Spreadsheet App. Tap on the header for column A to define the entire column with a single command.

In this case, we want to make column A display $(\text{Row})^2$, where Row is a variable representing the row number of each cell.

2. Press   to insert an equal sign. Press  to open the Vars menu and tap . Now tap **Spreadsheet**, **Numeric**, and select **Row**. Press  to square the Row variable and press  to see column A filled with the squares of the row numbers.
3. Scroll down through the column. What do you notice about the last digits of all the squares of the positive integers? Of the numerals 0-9, which ones can and cannot be the final digit of a square?



	A	B	C	D	E
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					



	A	B	C	D	E
1	1				
2	4				
3	9				
4	16				
5	25				
6	36				
7	49				
8	64				
9	81				
10	100				

3.1 Squares of integers must end in one of these digits:

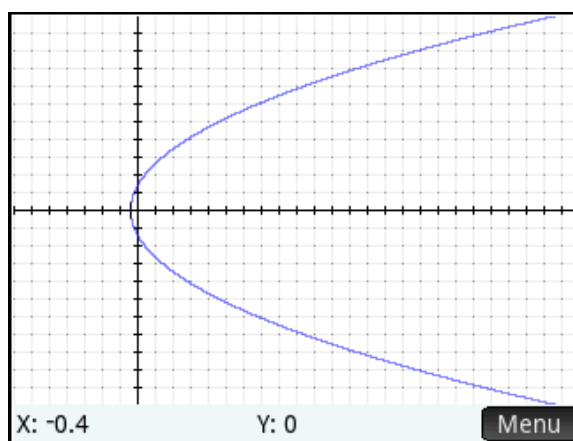
3.2 Squares of integers cannot end in these digits:

Suppose we start the equation of our parabola with $Y^2 = A \cdot X + B$. We know Y^2 can only end in certain digits, as discovered previously. If we can engineer $A \cdot X + B$ to only end in one or more of the other digits, then we can assure that our parabola has no integer-only coordinate pairs! Let's assume both A and B are single-digit integers and begin our search.

- Tap on the header for column B. Press **Shif** **=** to insert an equal sign. Enter a value for A, then press **x**. Press **Vars** to open the Vars menu and tap **App**. Now tap **Spreadsheet**, **Numeric**, and select **Row**. Press **+** and enter another single digit. Press **Enter** to see column B fill with values. If you need to, tap on the header for column B and tap **Edit** to change your values for A and B. In the figure to the right, $A=2$ and $B=3$ cannot be correct, as the numbers in column B can end in 1, 5, or 9 and these are all digits that the squares can end in as well.
- When you find the right values for A and B (there are two solutions!), press **Apps** to open the Apps Library and start the Advanced Graphing App. Enter your equation in V1 and press **Plot** to see the graph or press **Num** to see a table of values. Record your equation in the space below.

	A	B	C	D	E
1	1	5			
2	4	7			
3	9	9			
4	16	11			
5	25	13			
6	36	15			
7	49	17			
8	64	19			
9	81	21			
10	100	23			

=2*Row+3



X	Y	Y
0	-1.41421356237	1.41421356237
1	-2.64575131106	2.64575131106
2	-3.46410161514	3.46410161514
3	-4.12310562562	4.12310562562
4	-4.69041575982	4.69041575982
5	-5.19615242271	5.19615242271
6	-5.65685424949	5.65685424949
7	-6.0827625303	6.0827625303
8	-6.48074069841	6.48074069841
9	-6.8556546004	6.8556546004
0		

I found the second solution, too:

Teacher Notes for *Peculiar Parabolas*






This problem comes to us from *Challenging Problems in Algebra* by Alfred S. Posamentier and Charles T. Salking (Dover 1996). The problem involves some Number Theory. The challenge is to devise a parabola with the property that none of its points are such that both coordinates are integers. The technique is to let the left side of the equation be squares only (Y^2), so we know their last digits can only come from the set $\{0, 1, 4, 5, 6, 9\}$. Now we devise the right side to be a linear expression $A \cdot X + B$ so that the results have last digits that are from the set $\{2, 3, 7, 8\}$. This assures that Y^2 and $A \cdot X + B$ can never be equal if both X and Y are integers.

Answer Key

3.1 Squares of integers end in these digits: $\{0, 1, 4, 5, 6, 9\}$

3.2 Squares cannot end in these digits: $\{2, 3, 7, 8\}$

5. If X is an integer, then $5 \cdot X + 2$ always ends in 2 or 7 and $5 \cdot X + 3$ always ends in 3 or 8. This means our parabola can be $Y^2 = 5 \cdot X + 2$ or $Y^2 = 5 \cdot X + 3$.

To see the table in the last figure, press   to enter Numeric Setup and change **Num X Start** to 0 and **Num X Step** to 1. Now press  to open Numeric view. Tap  and select **Edge**. Tap  to see the table based on the x-values. Of course, the table is not a proof, but a verification tool for the investigative work by students in Question 5.


Extensions




1. Switch the roles of x and y to get two new equations with the same property.
2. Challenge your students with $5 \cdot X + 2$, $5 \cdot X + 7$, $5 \cdot X - 3$, etc. How many such parabolas are there?



Sicherman Dice

This probability activity explores what happens when you change the digits on a pair of dice. Like regular dice, Sicherman dice are cubical in shape and numbered with positive integers. Unlike regular dice, the first Sicherman die is numbered 1, 2, 2, 3, 3, and 4, while the second Sicherman die is numbered 1, 3, 4, 5, 6, 8. If two Sicherman dice are rolled repeatedly, what does the distribution of the sum of each roll look like?

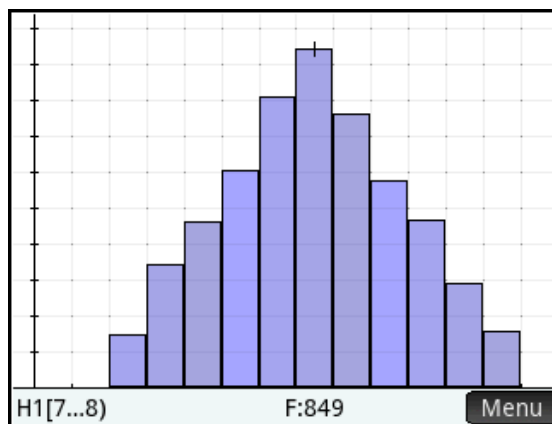
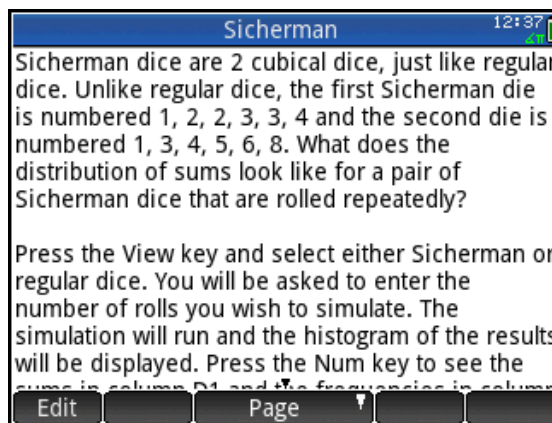
To simulate rolling regular and Sicherman dice, we created an app called *Sicherman*. Your instructor will send you the Sicherman App.

1. Press  and tap on the icon for the **Sicherman** app

The app is programmed to open in its Info view (more on this later). Press   (Info) at any time to return to this view. This view describes the app in general terms and directs the student to press  to get started.

2. Press  and select **Regular Dice**. You will be prompted to enter the number of rolls. Type 5000 and press .
3. The simulation will run and then the histogram of the sums of the rolls will be displayed.

As shown in the figure to the right, the distribution is symmetric and centered at 7, as expected.




4. Press  to see the data.

Column D1 contains the possible sums. Column D2 contains the frequencies corresponding to each sum in D1. Columns D3 and D4 represent the digits on the two dice used in the simulation.

	D1	D2	D3	D4
2	2	132	2	2
3	3	308	3	3
4	4	414	4	4
5	5	542	5	5
6	6	728	6	6
7	7	849		
8	8	684		
9	9	517		
10	10	421		
11	11	263		

849

5. Now press  and select **Sicherman Dice**. Run a simulation of 5000 rolls with these dice. Describe what you see in the data and the histogram.

6. Fill out the table of the sums below. The first row has the digits on one of the dice; the first column has the digits on the other dice. Each of the other cells in the table has the sum of its row and column header. The first row has been filled in for you.

	1	2	2	3	3	4
1	2	3	3	4	4	5
3						
4						
5						
6						
8						





7. Now use the table above to list the probabilities for each possible sum of 2-12. The probability of a 2 has been filled in for you.

Sum	2	3	4	5	6	7	8	9	10	11	12
P	1/36										

What do you notice about the probability distribution for the sum of a pair of Sicherman dice?

Teacher Notes for *Sicherman Dice*

Sicherman dice are named after their inventor, George Sicherman, and were originally reported in 1978 in response to the question of whether or not it was possible to devise a pair of dice that had the same probability distribution for their sums as standard dice. Like regular dice, the sum of the numbers on opposite sides are equal. For regular dice, the sum of opposite faces is 7. For Sicherman dice, the sum of opposite faces is 5 for the first die and 9 for the second. This is a great activity for introductory probability. I usually have several pairs of Sicherman dice (often marketed as Crazy Dice) available for students to play with.

Note: When students first launch this app, they MUST press   to enter the Program Library, then select the Sicherman program and tap  to open the program listing. They should then press  for Home view and continue with the simulation.

Answer Key

In Questions 1–4, students run a simulation of 5000 rolls of a pair of standard dice. The result is as expected: the distribution of the sums is very symmetrical, with 7 having the greatest frequency.

5. When the students repeat the simulation of 5000 rolls with a pair of Sicherman dice, they are usually quite surprised to see a very similar result: a symmetric distribution with 7 having the greatest frequency. How can this be?

6. Here is the table of possible sums.

	1	2	2	3	3	4
1	2	3	3	4	4	5
3	4	5	5	6	6	7
4	5	6	6	7	7	8
5	6	7	7	8	8	9
6	7	8	8	9	9	10
8	9	10	10	11	11	12

7. Here is the probability distribution. They fractions have been left unsimplified.






Sum	2	3	4	5	6	7	8	9	10	11	12
P	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Extensions




1. Go back to Numeric view (which contains the data from the Sicherman simulation) and compare the theoretical probabilities with the experimental probabilities. For example, in the figure for our simulation, the experimental probability of a sum of 7 is $849/5000$ or approximately 0.1698. The theoretical probability is $1/6$ or 0.1666... These two values are pretty close!

Conjecture and Proof: Midpoint Quadrilateral



The geometric construction that forms the basis of this activity consists of a quadrilateral ABCD. The midpoints of the sides of this quadrilateral are found and named KLMN. Then quadrilateral KLMN is constructed. We created an app named *MidpointQuad* that contains this construction. It is based on the built-in HP Prime Geometry app; as a result, the app uses the primary app keys as described in the table below.

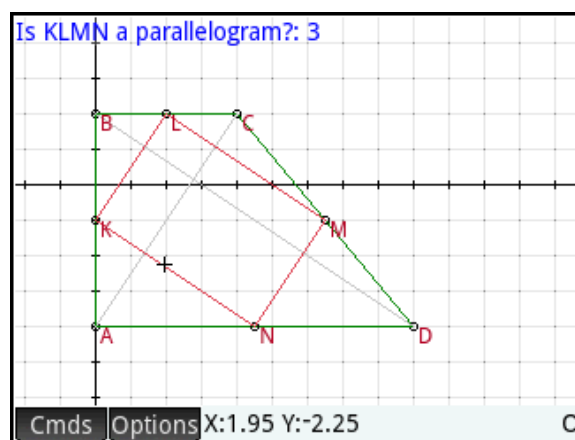
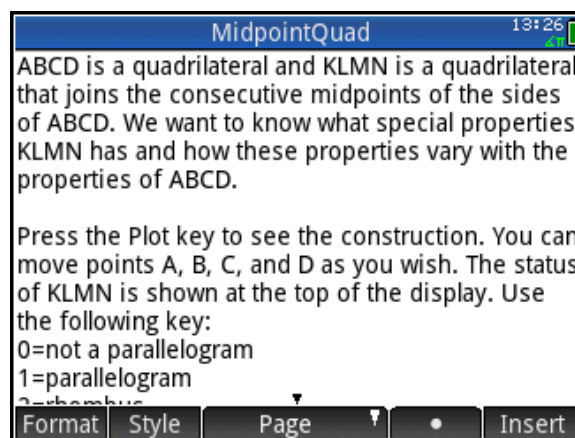
Key	View Name	Description
	App Library	Displays the library of apps
	Symbolic View	Displays the definitions of each geometric object
	Plot View	Displays the geometric construction
	Numeric View	Displays any calculations or measurements in the construction
	Info View	Displays a note explaining the purpose of the app and how to use it

Getting Started

1. Press  and tap on the icon for the **MidpointQuad** app. The app opens in Info view, with information about the app and how to use it. Press  (Info) to return to this view.
2. Press  to see the construction in Plot view.

The green quadrilateral ABCD is the basis of the construction. You can tap and drag to move any of its vertices or sides. The blue quadrilateral is formed by connecting the midpoints of the sides of ABCD. The diagonals AC and BD are also drawn to aid in conjecture and proof. At the top of the view is a test for whether or not KLMN is a parallelogram. There are 2 ways to move objects like points:

- Tap on the point, then drag your finger
- Tap on the point and press . Now use the cursor keys to move the point pixel by pixel. Press  to deselect the point.



The test can return any of the following five values:


- 0: KLMN is not a parallelogram
- 1: KLMN is a parallelogram
- 2: KLMN is a rhombus
- 3: KLMN is a rectangle
- 4: KLMN is a square

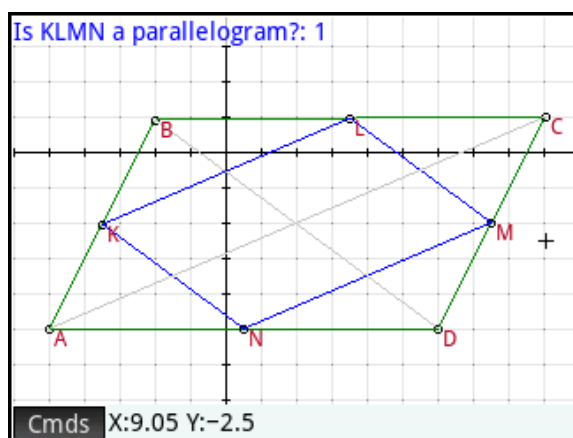
Take a moment and experiment with the construction by moving points A, B, C, or D. You will see that the test never returns zero. Even if you make ABCD concave, the test will continue to return a value of at least one. This leads us to our first conjecture, usually known as Varignon's Theorem:

The quadrilateral formed by the midpoints of the sides of any other quadrilateral is a parallelogram.

The Technique

Now let's make ABCD a parallelogram.

3. Move points A, B, C, and D so that they appear to form something close to a parallelogram. In the figure to the right, the opposite sides of ABCD appear to be roughly equal in length and parallel.
4. Press  to open Symbolic view of the app and observe the definitions of points A, B, C, and D.



We shall edit the coordinates of these points so that they are more exactly the coordinates of a parallelogram. Specifically, in our case (but not necessarily yours), we will make the following changes:

- GA: point(-5, -5)
- GB: point(-2, 1)
- GC: point(9, 1)
- GD: point(6, -5)

Object	Definition
<input checked="" type="checkbox"/> GA	point(-5.0, -5.0)
<input checked="" type="checkbox"/> GB	point(-2.0, 0.9)
<input checked="" type="checkbox"/> GC	point(9.05, 1.0)
<input checked="" type="checkbox"/> GD	point(6.0, -5.0)
<input checked="" type="checkbox"/> GE	quadrilateral(GA, GB, GC, GD)
<input checked="" type="checkbox"/> GG	segment(GA, GB)
<input checked="" type="checkbox"/> GH	segment(GB, GC)

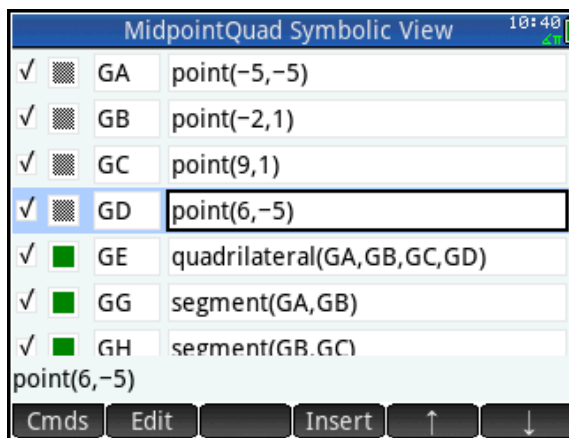
point(-5.0, -5.0)

Cmds Insert

5. Tap on the definition of GA to select it.
6. Tap **Edit** to edit the definition
7. Delete the current coordinates and enter your own. Press **Enter** or tap **OK** when you are done.

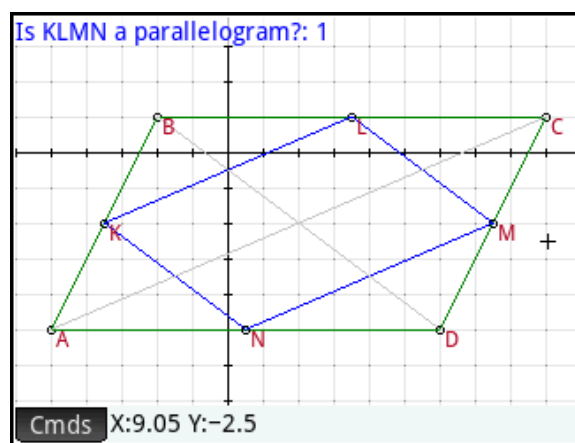
In the figure to the right, the editing has been done to the definitions of A, B, C, and D.

8. Press **Plot** to return to Plot view to see what has changed.



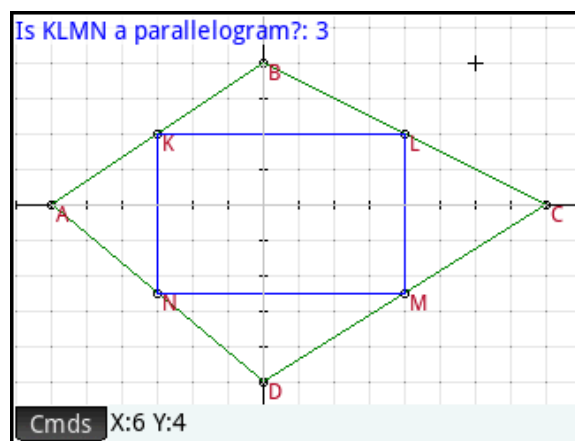
In this case, nothing has changed. Making ABCD a parallelogram has had no effect on KLMN. This example was selected merely to illustrate the technique involved:

- Move A, B, C, or D to form what appears to be a special quadrilateral
- Go to Symbolic view and edit the definitions to make the coordinates exact
- Return to Plot view to see the effects



Try it yourself

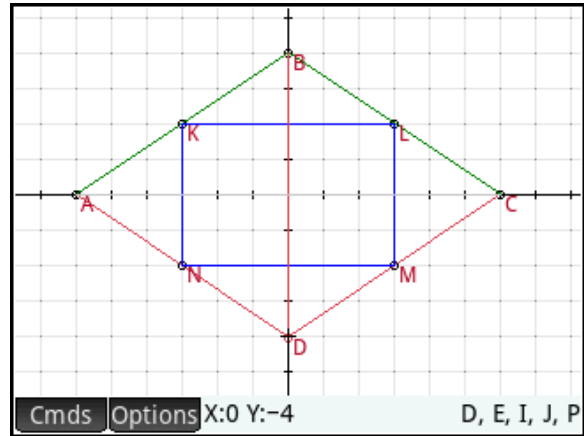
9. Make ABCD a kite. Once it is a kite, select one of the vertices and press **Enter** to select it. Now you can move the vertex carefully using the directional pad, one pixel at a time. Move the vertex so that ABCD continues to be a kite. In the figure to the right, you could move A or C horizontally.
10. Press **Esc** when you are done to deselect point A or C. Make a second conjecture based on your experience.



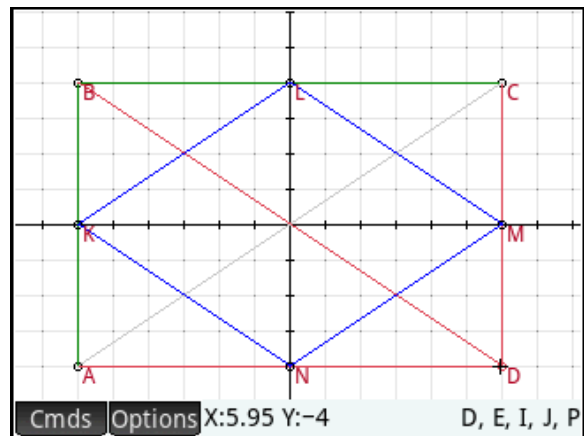
Do these on your own

Now that you have practiced the technique, try each of the following and make a conjecture for each one. In the figures for Questions 11-14, the test has been hidden.

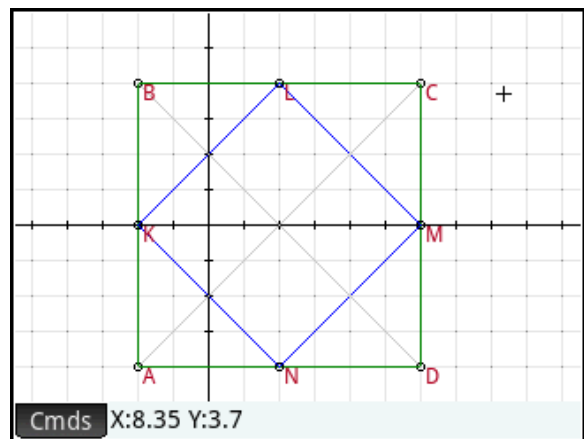
11. Make ABCD a rhombus. Make a conjecture about KLMN when ABCD is a rhombus.



12. Make ABCD a rectangle. Now make a conjecture about KLMN when ABCD is a rectangle.

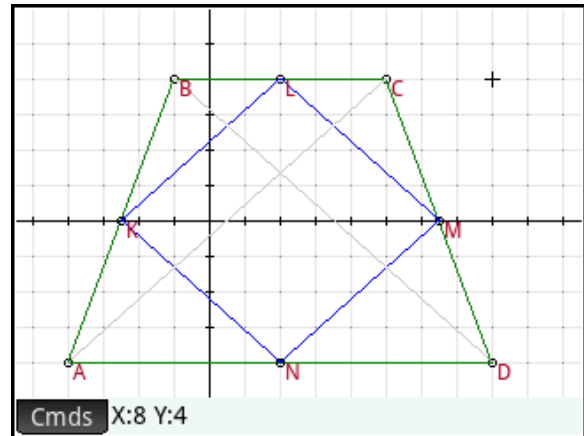


13. Make ABCD a square. Make a conjecture about KLMN when ABCD is a square.

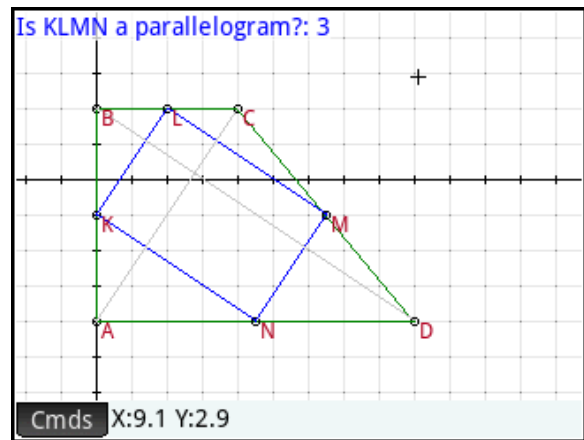


Extensions: Trapezoids are Interesting!

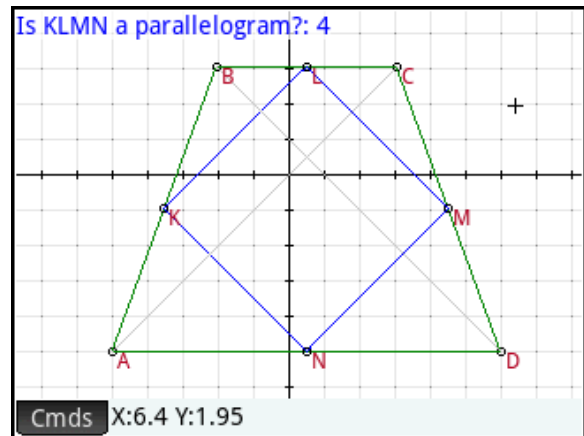
14. Make ABCD an isosceles trapezoid.
 Make a conjecture about KLMN when
 ABCD is an isosceles trapezoid.



15. You can go even further. In the figure to
 the right, ABCD is a special trapezoid
 and the test indicates that KLMN is a
 rectangle. What property or properties
 must trapezoid ABCD possess in order
 to make KLMN a rectangle?





16. In the figure to the right, ABCD is a very
 special isosceles trapezoid and the test
 indicates that KLMN is a square. What
 special property or properties must
 isosceles trapezoid ABCD possess in
 order to make KLMN a square?



Teacher Notes for Conjecture and Proof

In this set of activities, we used one simple construction to generate a number of conjectures to prove. The Numeric, Symbolic, and Plot views of the HP Prime Geometry app encourage open-ended experimentation, the making of conjectures, and the exploration of paths to proofs. The vision is to have the class share this construction, and divide into groups to pursue group conjectures and proofs. As the students own their conjectures, devising convincing proofs becomes more interesting with an authentic problem. Rather than have the students create this construction themselves (which is entirely possible but not done here to save time), we created an app named *MidpointQuad*, based on the built-in HP Prime Geometry app. Most of the proofs rely on the perpendicularity or congruence of the diagonals of the original quadrilateral. Others rest on the perpendicularity of the sides. One is based on combining two previous proofs.

For Questions 11-14, the test results are hidden in Plot view. To show/hide tests and measurements, press  to go to Numeric view, tap on the check box to select the test, and tap again to toggle it to unchecked (or tap ). Numeric view is used to create tests, measurements, and other calculations. Those that are checked appear in Plot view.

Answer Key

10 If ABCD is a kite, then KLMN is a rectangle. This follows from the kite having perpendicular diagonals. We already know KLMN is a parallelogram; we seek to prove it has a right angle. In $\triangle ABC$, KL is parallel to AC. In $\triangle BCD$, LM is parallel to BD. But $AC \perp BD$, so it must be that $KL \perp LM$

11. If ABCD is a rhombus, then KLMN is a rectangle. Again, this follows from the rhombus having perpendicular diagonals. The proof is the same as for the case where ABCD is a kite!

12. If ABCD is a rectangle, then KLMN is a rhombus. This follows from the right angles in ABCD. We already know that KLMN is a parallelogram; we seek to prove two consecutive sides are equal in length. Show that $\triangle KBL \cong \triangle MCL$ (SSS); this means $KL=LM$.

13. If ABCD is a square, then so is KLMN. Just put together the two arguments in #11 and #12! Since ABCD is both a rhombus and a rectangle, then KLMN is both as well!

Extensions

14. If ABCD is an isosceles trapezoid, then KLMN is a rhombus. From Varignon's Theorem, we already know KLMN is a parallelogram; we seek to prove it has two consecutive sides equal in length. In $\triangle ABC$, KL is half the length of AC. Likewise, in $\triangle BCD$, LM is half the length of BD. But in an isosceles trapezoid, the diagonals AC and BD are equal in length; thus so are KL and LM. But a parallelogram with equal consecutive sides is a rhombus.

15. ABCD has two consecutive right angles and the lengths of the sides AB, BC, and AD were chosen so that $\triangle LBK \sim \triangle KAN$. So $\angle BLK$ and $\angle BKL$ are complementary because they are the acute angles in right $\triangle BLK$. Because of the similarity, $\angle BLK = \angle AKN$. So now $\angle AKN$ and $\angle BKL$ are complementary. This makes $\angle LKN$ a right angle. Also note that the diagonals are perpendicular.

16. ABCD has perpendicular diagonals. We already know from #14 that KLMN is a rhombus. We seek to prove it has 2 sides that are perpendicular. In $\triangle ABC$, KL is parallel to AC; likewise, in $\triangle BCD$, LM is parallel to BD. But if $AC \perp BD$, then $KL \perp LM$.

Cyclic and Bicentric Quadrilaterals

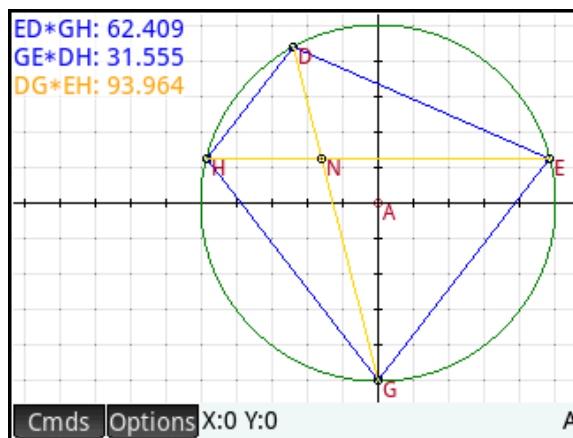
Part 1: Cyclic Quadrilaterals

The instructor will send you an HP Prime app called *CyclicQuad* for this part of the activity. A cyclic quadrilateral is a convex quadrilateral that has a circumscribed circle.

1. Press  to open the App Library and select the **CyclicQuad** app.

The construction consists of DEGH, a cyclic quadrilateral circumscribed by circle A.

2. Tap and drag any of the points D, E, G, or H to change the quadrilateral. Which of the following can DEGH never be?
 - Square
 - Rhombus (non-square)
 - Rectangle (non-square)
 - Parallelogram (non-rhombus)
 - Isosceles trapezoid
 - Kite







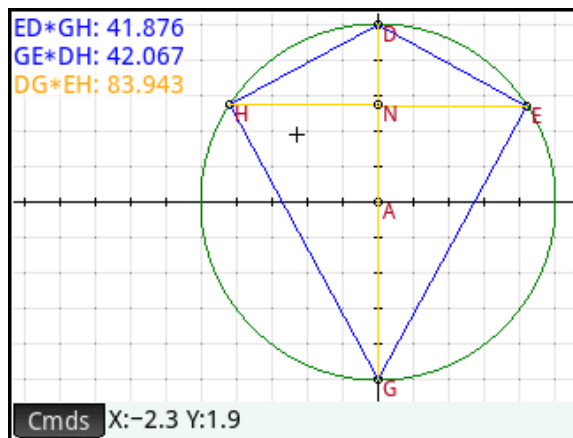
A cyclic quadrilateral cannot be:

Just move the points of the quadrilateral around enough to convince yourself for each one.

3. Notice $\sphericalangle HDE$ and $\sphericalangle HGE$ are both inscribed angles that subtend the entirety of the circle; likewise with $\sphericalangle DHG$ and $\sphericalangle DEG$. This leads us to a defining characteristic of cyclic quadrilaterals. Make a conjecture.

A quadrilateral is cyclic if and only if...

4. Make DEGH into a kite, similar to that shown to the right. Tap segment HE and press  to select it. Now use  and  to move the diagonal vertically. What do you notice about the angles of this kite? Press  to deselect segment DG. Make a conjecture to prove later:



If a cyclic quadrilateral is a kite,

then _____

5. Notice that we have defined the following products as well:
- The product of the opposite side lengths ED and GH
 - The product of the opposite side lengths GE and DH
 - The product of the lengths of the diagonals DG and EH

What do you notice about these products?
Make a conjecture:

6. This conjecture is known as Ptolemy's Theorem. If DEGH is a rectangle, to what theorem does it become equivalent?

There is also a famous formula for the area of a cyclic quadrilateral. This formula looks a bit like Heron's formula for the area of a triangle. It is called Brahmagupta's Formula:

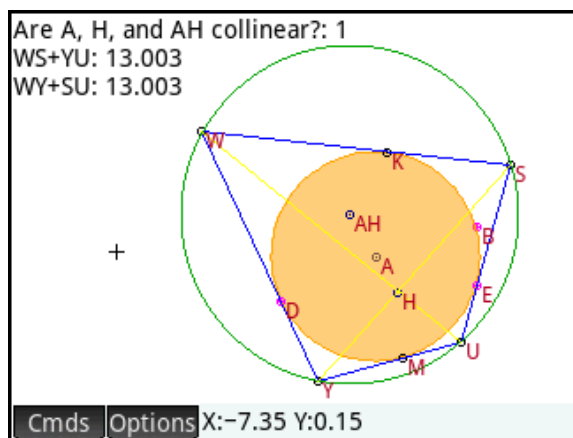
$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$, where s is the semi-perimeter of the quadrilateral and $a, b, c,$ and d are the side lengths.

Part 2: Bicentric Quadrilaterals

The instructor will send you an app named *BicentricQuad* for this part of the activity. A bicentric quadrilateral is one that has both an inscribed circle and a circumscribed circle. In other words, it is a cyclic quadrilateral that has an inscribed circle as well.

1. Press  to open the App Library and select the **BicentricQuad** app.

In the construction, quadrilateral $WSUY$ is a bicentric quadrilateral. Point A is the center of the inscribed circle and point AH is the center of the circumscribed circle. Point H is the intersection of the diagonals. You can move points D , H , and E , but not K and M .

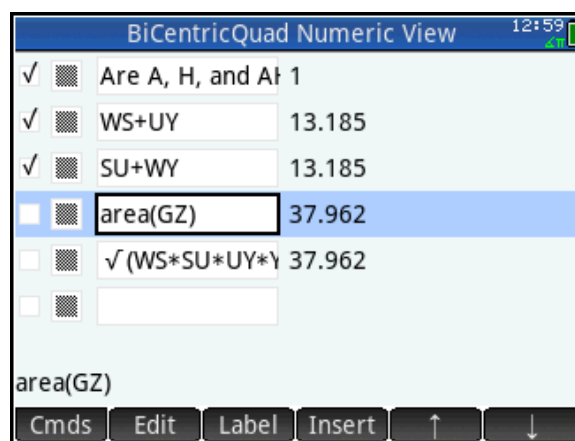
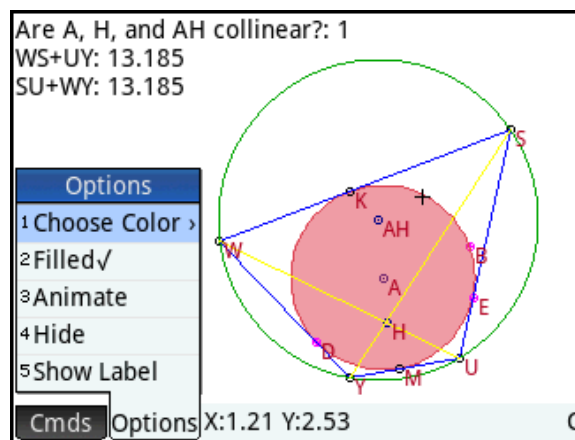


2. In Part 1, you discovered that cyclic quadrilaterals can be right kites, squares, rectangles, and isosceles trapezoids. Which type of cyclic quadrilateral cannot be bicentric? Why?

3. The sums of the lengths of opposite sides are also displayed. What do you notice about these two sums? Make a conjecture.

4. To support your conjecture, notice point W is a point exterior to circle A , with two tangent segments, \overline{WK} and \overline{WD} , of equal length. The same is true of the other three vertices of quadrilateral $WSUY$. You can now write four equations and manipulate them to get the desired result.

5. Tap on Circle A and then tap **Options**.
From the pop-up menu, tap **Filled** to deselect the option.
6. Press **Num** to open Numeric view. You will see that there are two items at the end of the list that are not selected; that is, they do not have check marks to their left. One is the area of the quadrilateral WSUY; the other is the square root of the product of the side lengths of WSUY. Check them both so that they appear in Plot view by selecting each one in turn and tapping **✓**.
7. Press **Plot** to return to Plot view and move points H, E, or D. What do you notice about these two measurements? Make a conjecture.

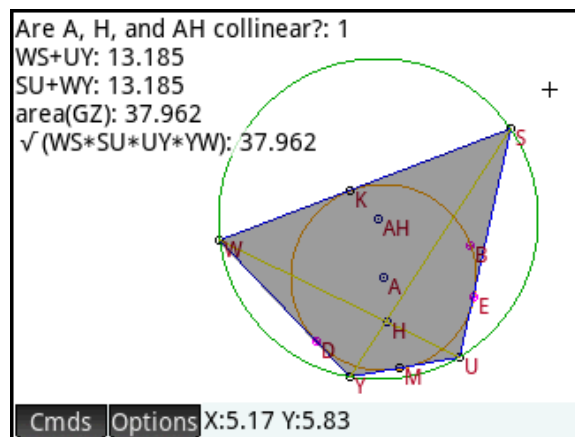


In the first part of this activity, we ended with Brahmagupta's Formula:

$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$, where s is the semi-perimeter of the quadrilateral and $a, b, c,$ and d are the side lengths.

Since the sums of the opposite sides are equal, the semi-perimeter, s , is equal to either of these sums: $a+c$ or $b+d$.

8. Use this fact to simplify Brahmagupta's Formula to the conjecture you made in #6.



Finally, we note that the question at the top of Plot view always has a 1 after it. This is a test that returns 0 for false and 1 for true, so it seems the incenter, the circumcenter, and the intersection of the diagonals in a bicentric quadrilateral are always collinear.

Teacher Notes for Cyclic and Bicentric Quadrilaterals

This activity affords students the opportunity to make authentic conjectures and grapple with their proofs. The dynamic constructions are given to them via two HP Prime Apps.

Answer Key

Part 1: Cyclic Quadrilaterals

2. A cyclic quadrilateral cannot be only a parallelogram or only a rhombus.
3. A quadrilateral is cyclic if and only if its opposite angles are supplementary. This is why neither a proper parallelogram nor a proper rhombus can be a cyclic quadrilateral, as their opposite angles are equal and can only be supplementary if they are right. But if they are right angles, then they are no longer either a proper parallelogram nor a proper rhombus (but a rectangle or square).
4. If a cyclic quadrilateral is a kite, then it has 2 right angles.
5. The sum of the products of the lengths of opposite sides of a cyclic quadrilateral is equal to the product of the lengths of the diagonals.
6. It becomes equivalent to the Theorem of Pythagorus.

Part 2: Bicentric Quadrilaterals

2. A rectangle can be cyclic but cannot be bicentric.
3. The sum of the lengths of opposite sides of a cyclic quadrilateral are equal.
4. From the fact that tangent segments from an exterior point are equal in length, we have:
1) $WK = WD$, 2) $SK = SE$, 3) $UM = UE$, and 4) $YD = YM$.
From 1) and 2) we have $WK + KS = WD + SE$ or $WS = WD + SE$
From 1) and 4) we have $WD + DY = WK + YM$ or $WY = WK + YM$
Likewise, 3) and 4) give us $YM + MU = YD + UE$ or $YU = YD + UE$
Finally, 2) and 3) give us $SE + EU = SK + UM$ or $SU = SK + UM$
So $WS + YU = WD + SE + YD + UE$ or $WS + YU = WY + SU$
7. The area of a bicentric quadrilateral is equal to the square root of the product of the lengths of its sides.

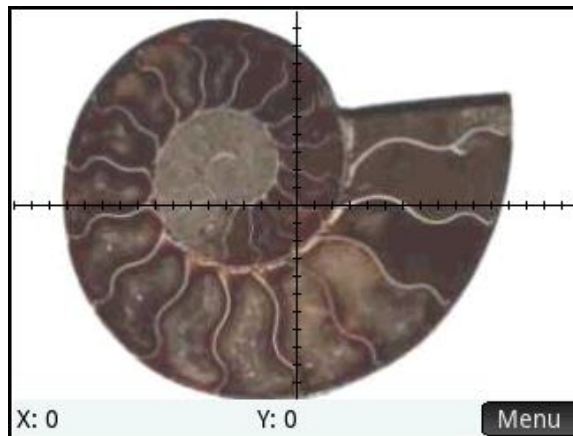
$$8. A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

$$A = \sqrt{(a+c-a)(b+d-b)(a+c-c)(b+d-d)} \text{ or } A = \sqrt{c*d*a*b}$$

A Nautilus Shell

In this activity, we fit a logarithmic spiral to a photo of a cross section of a fossil! Your instructor will send you the HP Prime App named *NautilusShell*.

1. Press **Apps** to open the Apps Library and start the app **NautilusShell**.
2. Press **Plot** and tap **OK** to see the photo of a cross section of a fossil, as shown to the right.



The fossil actually measures 10.5 cm across at its widest. Notice the smooth spiral that marks the growth of this ancient creature. We wish to fit a mathematical polar curve to this spiral.

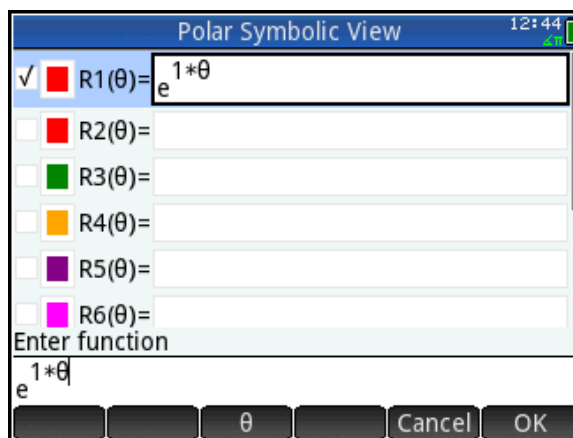
3. Your first task is to resize and move the axes so that the origin is conveniently located and the width of the fossil actually measures 10.5 units along the x-axis. Pinch diagonally to keep the units the same on both axes. When you are as close as you can get, record the domain and range of your window.

X runs from approximately _____ to _____ and Y runs from approximately _____ to _____.

The mathematical curve we wish to use to approximate the spiral is called a logarithmic spiral. It is basically the exponential curve $Y = e^{bX}$ in polar form:

$$R = e^{b\theta} \text{ or } R = e^{\frac{\theta}{b}}$$

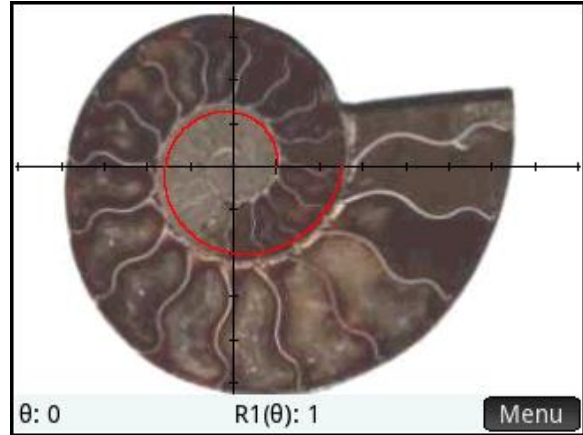
4. Press **Symb** to return to Symbolic view and enter a definition in R1(θ). You will have to find the value of b that best approximates the spiral. In the figure to the right, we started with $b=1$. Record your equation below.



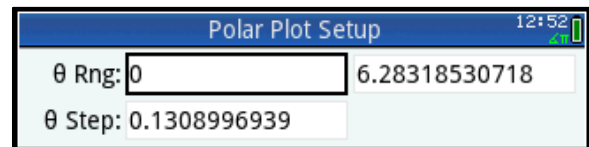
R1(θ) = _____

Note: Be sure to choose a color for your spiral! Tap on the color picker to the left of R1(θ).

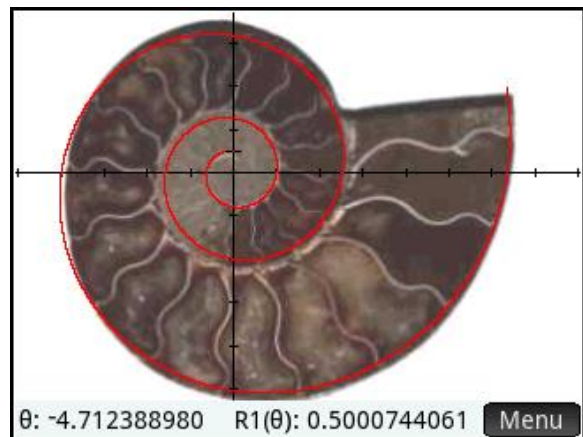
Once you have the best fit you can devise, you will need to adjust the start and end values for θ so that the entire spiral is traced.



- Press **Shift** **Plot Setup** to view Plot Setup. Here, adjust the two values in **θ Rng** until the spiral starts and stops where you want it to. Record your θ –range in the space below.



θ runs from _____ to _____.



And there you have it: a simple mathematical expression in two variables approximates the growth spiral of a creature that lived millions of years ago – and many that are alive today as well!

Teacher Notes for *A Nautilus Shell*

This is a great activity for introducing students to polar equations. You can find these fossils in any rock shop and take your own photo. I usually pass my fossil around so students can pick it up and look at the real thing carefully.

Answer Key

3. X runs from approximately -5 to 8 while Y runs from -5 to approximately 3.6. Answers will vary here and that is OK.

4. In our figures, $R1(\theta) = \theta/6.8$ or $R1(\theta) = 0.147*\theta$. Answers will vary here as well.

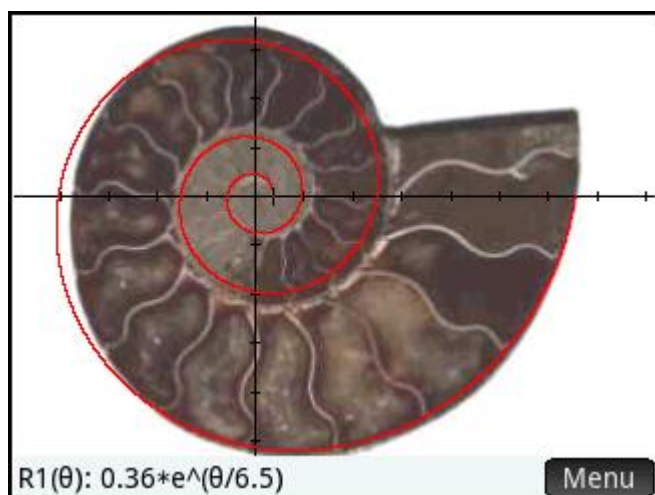
5. In our figures, θ runs from $-3\pi/2$ to approximately 4.1π . Answers will vary here as well.

Extensions

1. Where one places the origin has a significant effect on the equation of the spiral. Attempting to make the spiral start on the positive ray of the x-axis actually complicates the curve-fitting process. See the figure to the right and note the equations is now

$$R1(\theta) = a * e^{b\theta}$$

Here, θ runs from 0 to 6π .



2. Since we started by saying the logarithmic spiral is the polar equivalent of the exponential curve $y=e^x$, it is natural to ask what the polar equivalent of the linear function $y=ax+b$ is and what its graph looks like.
3. Continue this reasoning with even simpler expressions like $R = \theta$ and $R=C$ where C is a constant.

Pascal's Triangle

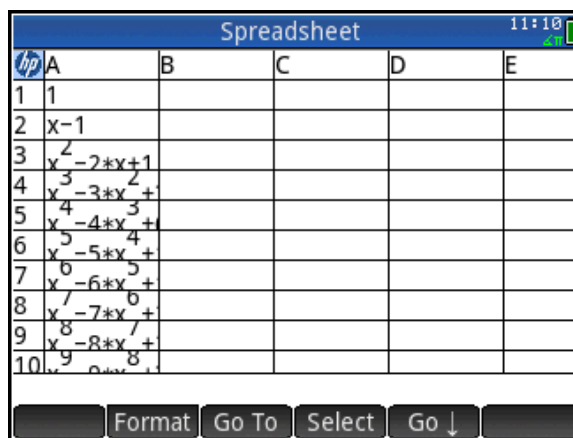
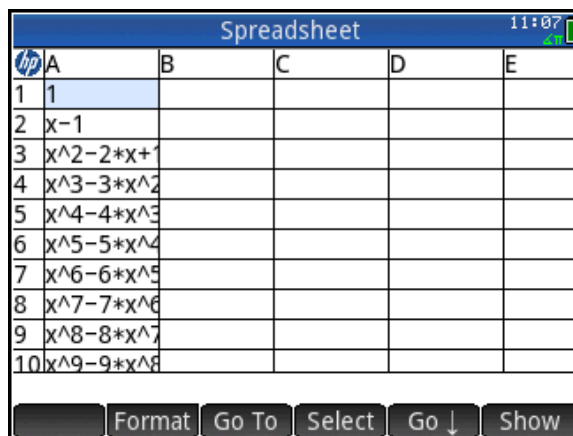
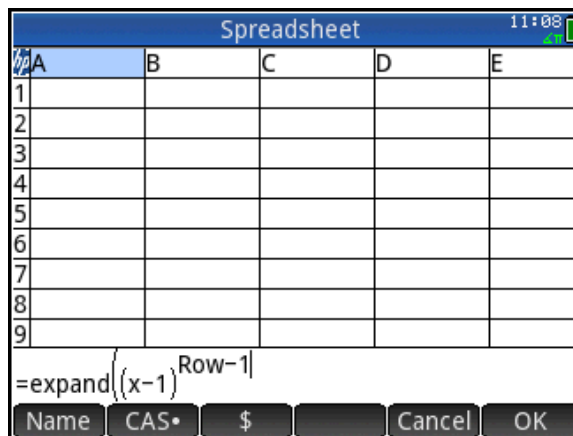
In this activity, we explore Pascal's Triangle and its application to the expansion of $(X-1)^N$ for $N=0, 1, 2, 3, \dots$

1. Press **Apps** and start the Spreadsheet App.

We will define Column A to be the expansion of $(x-1)^{n-1}$, where n is the row number of each cell in Column A.

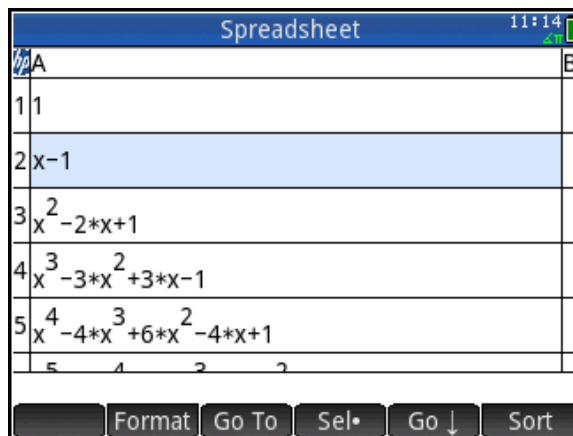
2. Tap on the header for Column A. Press **Shift** **=** (=) to enter an equal sign to start our column definition. Press **Mem B**, tap **CAS**, then tap **Algebra** and select **Expand**. Press **()** to enter a pair of parentheses. Press **ALPHA** **Shift** **x** to enter a lowercase x, then **-** **1** to complete the binomial. Press **▶** to move past the right parenthesis. Press **x^y** to start the exponent. Here we will use the variable *Row* instead of n . Press **Vars** to open the Variables menu. Tap **App**, then tap **Spreadsheet, Numeric**, and select **Row**. Press **-** **1** to complete the exponent. Tap **CAS** to use the CAS to evaluate the expression. Now press **Enter** to see the result.

3. Let's make the results more readable. Tap the hp logo in the top left corner. This selects everything. Tap **Format**, scroll down to **Textbook**, and select **Yes**. Now our expressions look readable, but we must increase both the height and width of the cells in Column A.

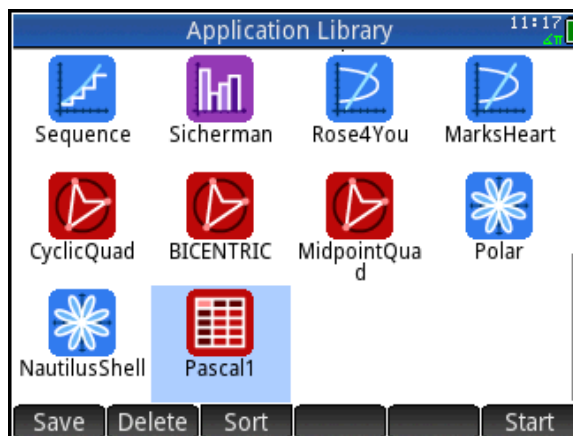


- With the hp logo still selected, tap **Format** again, select **Row** \updownarrow and change the value to 30. Now tap to select a cell in Column A and pinch to increase the width of the cell. The finished product is shown to the right. This is quite a nice visualization for students to explore.

You can save this app for future use and send it to your students.

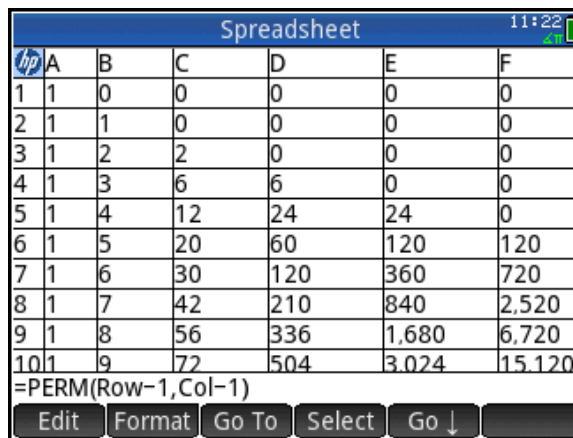


- Press **Apps Info** to open the Apps Library. The Spreadsheet App is shown as selected. Tap **Save** and enter a name you will remember. We will use *Pascal1*. Press **Enter** twice and you will see your new app in the library.



We will now create another visualization of Pascal's Triangle.

- Scroll up to the Spreadsheet App and reset it. Start it and tap on the hp logo to select the entire spreadsheet.
- Press **Shift** **=** (=) to enter an equal sign to start our column definition. Press **Mem B**, tap **Math**, then tap **Probability** and select **Permutation**. Enter *Row - 1*, then press **Eval** to enter a comma. Finally, enter *Col - 1*. Remember that *Row* (and *Col*) can be found by pressing **Vars**, tapping **App**, then tapping **Spreadsheet** and **Numeric**. Press **Enter** to see just the coefficients this time.



The combination of the Spreadsheet App and the CAS allows a whole new world of visualizations to be easily created and saved for future use.

Teacher Notes for Pascal's Triangle

This activity is a teacher activity rather than a student activity. There are two purposes to this activity:

1. To illustrate the power of the app variables *Row* and *Col* (also see *Cell*)
2. To illustrate the power of invoking the CAS from inside the Spreadsheet App
3. To encourage teachers to think about new visualizations of exact numeric or symbolic patterns that students can explore and learn from

We used Pascal's Triangle as an example to illustrate the techniques above. The benefit to students is that they have the opportunity to explore patterns and make conjectures from the very beginning. The teacher can then build on student discoveries and engage in authentic mathematical discourse.

Extensions and Variations

1. Define Column A using $=\sqrt{Row}$ and invoke the CAS. Students will then have a visualization of the square roots of the counting numbers. They can observe that there are three cases:
 - i. The square root resolves to an integer, as in $\sqrt{4} = 2$
 - ii. The square root does not resolve at all, as in $\sqrt{5}$
 - iii. The square root resolves partially, as in $\sqrt{8} = 2\sqrt{2}$

	A	B	C	D	E
1	1				
2	$\sqrt{2}$				
3	$\sqrt{3}$				
4	2				
5	$\sqrt{5}$				
6	$\sqrt{6}$				
7	$\sqrt{7}$				
8	$2*\sqrt{2}$				
9	3				
10	$\sqrt{10}$				

Formula bar: $=\sqrt{Row}$

2. Define Column A using $=\text{simplify}(Row*\pi/6)$. Then define Column B using $=\text{COS}\left(\frac{Row*\pi}{6}\right)$ and Column C using $=\text{SIN}\left(\frac{Row*\pi}{6}\right)$ and invoke the CAS for all three. Students then have an exact table of basic trigonometric values for these two functions.

	Angle	Cosine	Sine	D	E
1	$\pi/6$	$\sqrt{3}/2$	$1/2$		
2	$\pi/3$	$1/2$	$\sqrt{3}/2$		
3	$\pi/2$	0	1		
4	$2*\pi/3$	$-(1/2)$	$\sqrt{3}/2$		
5	$5*\pi/6$	$-\sqrt{3}/2$	$1/2$		
6	π	-1	0		
7	$7*\pi/6$	$-\sqrt{3}/2$	$-(1/2)$		
8	$4*\pi/3$	$-(1/2)$	$-\sqrt{3}/2$		
9	$3*\pi/2$	0	-1		
10	$5*\pi/3$	$1/2$	$-\sqrt{3}/2$		


Formula bar: $\text{SIN}((Row*\pi/6))$

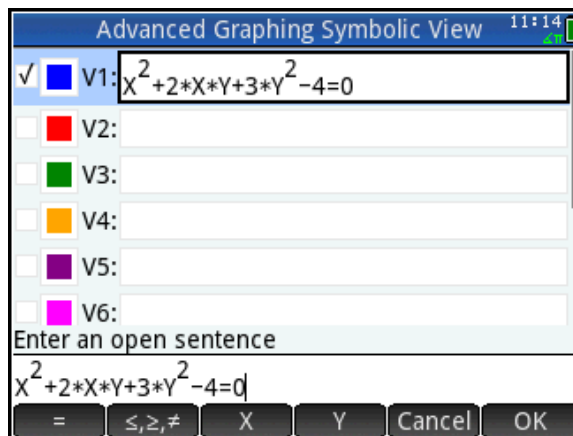
Note: to give a column a name, tap **Format** and select **Name**.


All of these visualizations can be saved and used whenever needed!

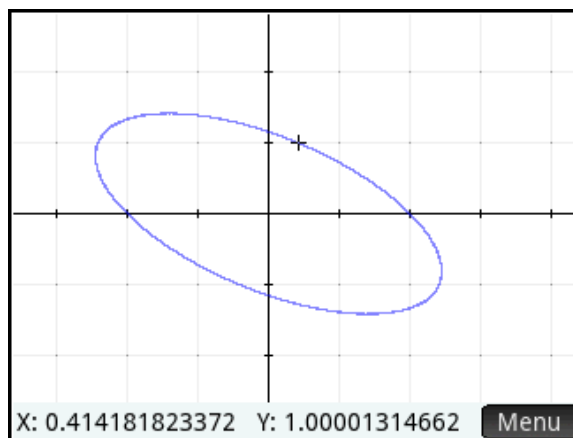
Rotation of Axes

In this activity, we examine the graph of $x^2 + 2xy + 3y^2 - 4 = 0$.

1. Press  to open the Apps Library and start the Advanced Graphing App.
2. Enter the equation for our ellipse above in V1.



3. Press  to see the graph of a rotated conic. Pinch to zoom in and drag to re-center the graph, as shown in the figure to the right.
4. By what angle would we need to rotate this conic in order to eliminate the xy -term?

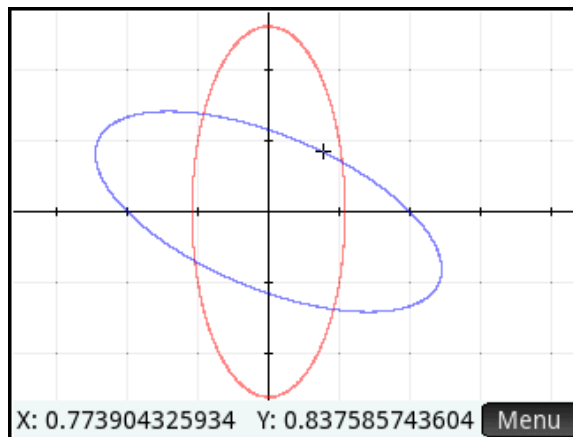


$\Theta =$

5. Find the sine and cosine of your angle using the CAS.

$\sin(\theta) =$

$\cos(\theta) =$

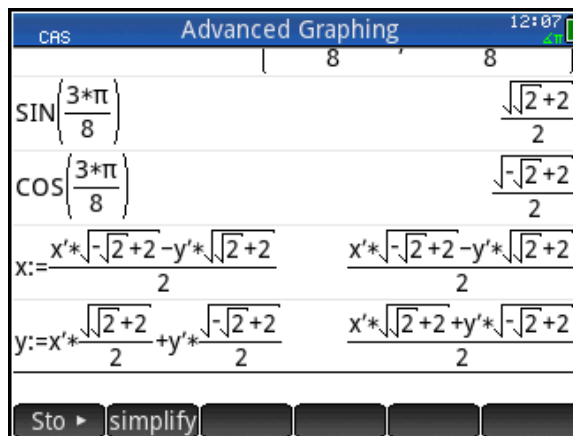


6. Return to Symbolic view and define V2 to be the rotation of V1 through your angle. Specifically, define V2 to be

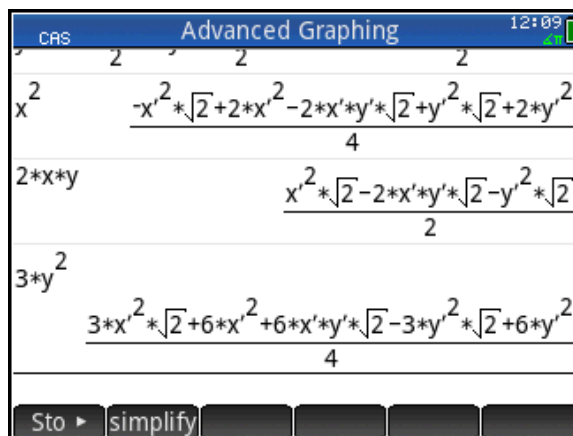
$$V1(X * \cos(\theta) - Y * \sin(\theta), X * \sin(\theta) + Y * \cos(\theta))$$

You can see by visual inspection that our angle of rotation appears to be correct. Now we want to verify that this rotation did, indeed remove the xy -term from our symbolic definition.

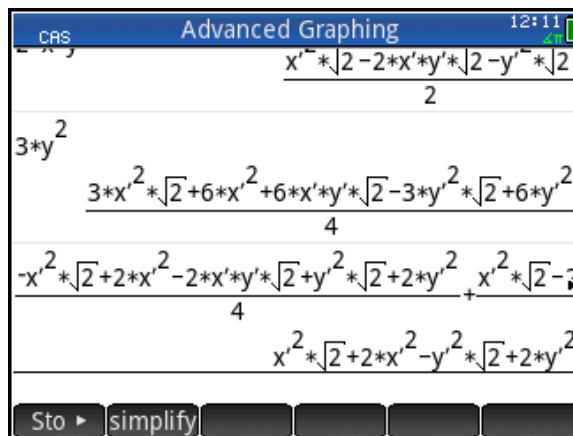
7. In the CAS, define our x and y in terms of the exact values of $\sin(\theta)$ and $\cos(\theta)$.




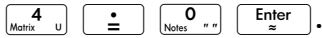
8. Now evaluate x^2 , $2xy$, and $3y^2$. Each of these has an $x'y'$ -term. What do you notice about these three terms?



9. Add the three terms together. Double-tap on the first, then press $\boxed{+}$. Double-tap on the second, then press $\boxed{+}$ again. Finally, double-tap on third and press $\boxed{\text{Enter}}$. Does the sum agree with your answer to #8 above?



10. Complete the definition. Press 



CAS Advanced Graphing 12:21

$$\frac{3x'^2 \sqrt{2+6x'^2} + 6x'y' \sqrt{2-3y'^2} + 6y'^2 \sqrt{2+6y'^2}}{4}$$


$$\frac{-x'^2 \sqrt{2+2x'^2} - 2x'y' \sqrt{2+y'^2} + y'^2 \sqrt{2+2y'^2} + x'^2 \sqrt{2-2x'^2} - 2x'y' \sqrt{2-x'^2} - y'^2 \sqrt{2-2y'^2}}{4}$$

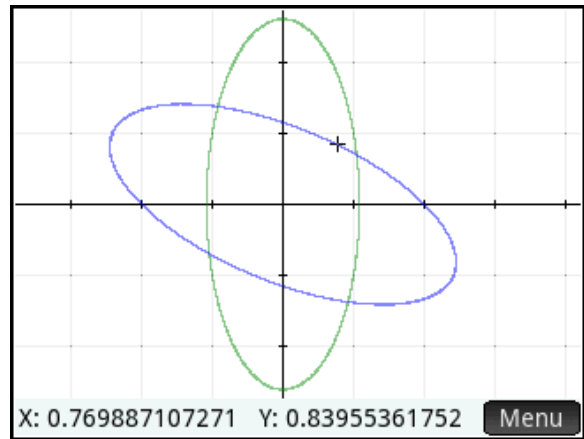
$$x'^2 \sqrt{2+2x'^2} - y'^2 \sqrt{2+2y'^2} - 4 = 0$$

$$x'^2 \sqrt{2+2x'^2} - y'^2 \sqrt{2+2y'^2} - 4 = 0$$

Sto simplify

11. Enter our definition in V3. You can use copy and paste, but you will have to change x' to X and y' to Y.

12. Uncheck V2 and press  to see the graph of our rotated conic without the XY-term.



Teacher Notes for Rotation of Axes

Rotating axes to eliminate the xy -term in a general conic may or may not be an important skill in the future; HP Prime can easily graph these general conics. It is included here to show how the CAS and Advanced Graphing App can work together to give multiple views on the process. From the graphical side, defining V2 as a rotation of V1 gives students an easy and quick way to see that the rotation works and that they have the correct angle. From the symbolic perspective, the CAS lets the students define x and y in terms of x' and y' , then follow step by step to define the conic in terms of x' and y' , and literally watch the $x'y'$ -terms drop out.

Answer Key

4. The angle of rotation is given by

$$\cot(2\theta) = \frac{1-3}{2}. \text{ The positive principle value}$$

$$\text{is } \theta = \frac{3\pi}{8}.$$

$$5. \sin\left(\frac{3\pi}{8}\right) = \frac{\sqrt{\sqrt{2}+2}}{2} \text{ and}$$

$$\cos\left(\frac{3\pi}{8}\right) = \frac{\sqrt{-\sqrt{2}+2}}{2}$$

6. The definition of V2 is shown in the figure to the right. Once could just as easily use $\sin(3\pi/8)$ and $\cos(3\pi/8)$ in the definition.

8. The three terms sum to zero.



9. Yes

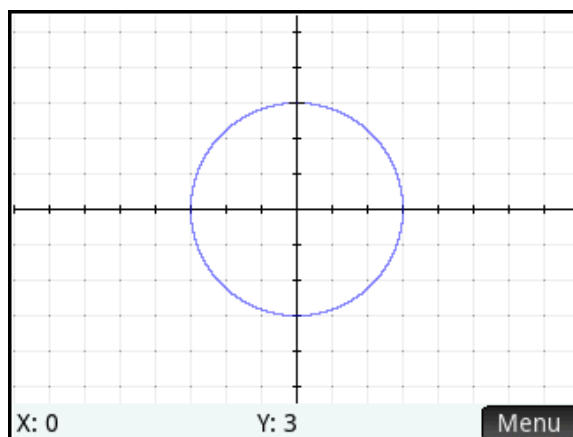
10-12. The definition of V3 is also shown in the figure to the right. V2 and V3 produce the same graph.


Graphs Polar and Cartesian

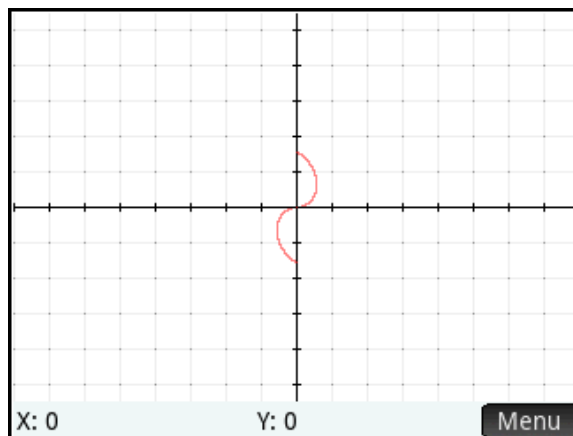
Conversion between polar and Cartesian coordinates is usually done by means of the equations $r = \sqrt{x^2 + y^2}$ and $\theta = a \tan\left(\frac{y}{x}\right)$. We make use of these equations in this activity to draw circles, spirals and roses using the Advanced Graphing App.


Let's start with a simple example. $r=3$ is a polar equation whose graph is a circle with radius 3, centered at the origin.

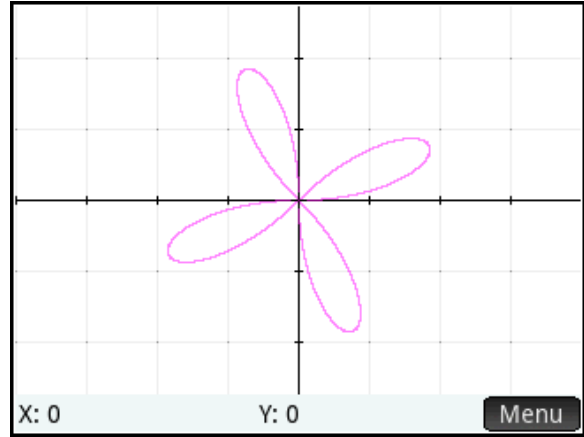
1. Use the equations above to get an equation in X and Y.
2. Press  to open the Apps Library and start the Advanced Graphing App. Enter your equation in V1.
3. Press  to see the graph. Is it the graph you expected?



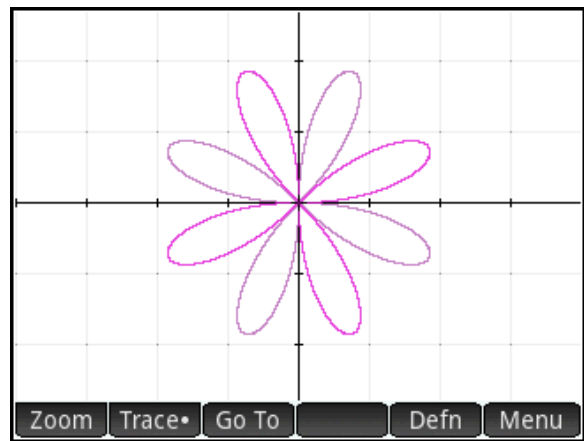
4. Now repeat Steps 1-3 for the spiral $r=\theta$. Enter your equation in V2 and uncheck V1 so it will not be plotted.
5. It appears as if part of the graph is plotted, but not the entire spiral. Take the tangent of both sides of your equation and simplify the result. Enter this new equation in V3 in Symbolic view. Now press  to see the graphs of both V2 and V3. What do you see?



6. Now try the same techniques with the polar rose $r=2*\sin(4*\theta)$. First, convert it to an equation in X and Y. Then enter the equation as V4. Uncheck V2 and V3 so they will not be plotted. Press  to see the graph. Our graph has 4 petals, but it is supposed to have 8.



7. Can you transform V4 to plot the other 4 petals?



Teacher Notes for Graphs Polar and Cartesian

This activity returns to the fundamental notions of equivalence and conditions we examined earlier, but here it is at a much more advanced level.

1. V1 is $\sqrt{X^2 + Y^2} = 3$. The graph is shown at the right.

3. Yes, the graph is a circle as expected, with radius 3 and center at the origin.

4. The equation of V2 is

$$\sqrt{X^2 + Y^2} = a \tan\left(\frac{Y}{X}\right).$$

5. The equation in V3 is $\tan(\sqrt{X^2 + Y^2}) = \frac{Y}{X}$

The graph is a double Archimedean spiral.

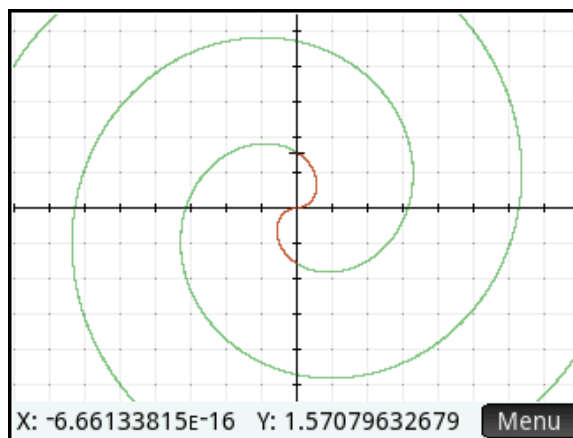
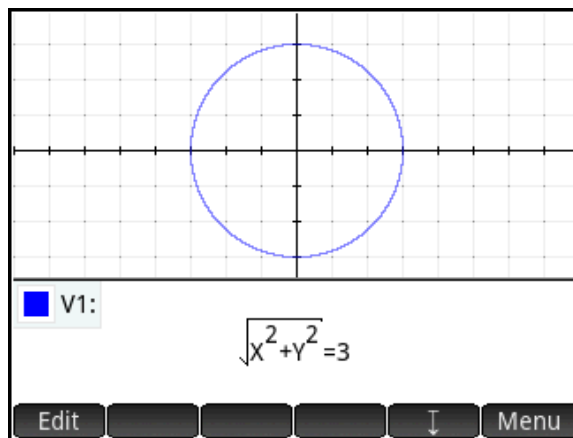
The left side of V2 can only return non-negative real numbers. The right side can only return real numbers between $-\pi/2$ and $\pi/2$. Thus, the V2 equation can only be true when both sides return numbers in the interval $[0, \pi/2)$. But the right side can only return those non-negative values when Y/X is positive; that is, when both X and Y have the same sign.

Thus the graph only appears in the 1st and 3rd Quadrants and only in the first loop. After that

$\sqrt{X^2 + Y^2} \geq \frac{\pi}{2}$. However, once you take the tangent of both sides, the result is

$\tan(\sqrt{X^2 + Y^2}) = \frac{Y}{X}$. Both sides of this equation can return any real number you like. Nor are X

and Y required to have the same sign. Notice, though, that $X \neq 0$, so there are no y -intercepts. Every apparent y -intercept is a removable discontinuity.

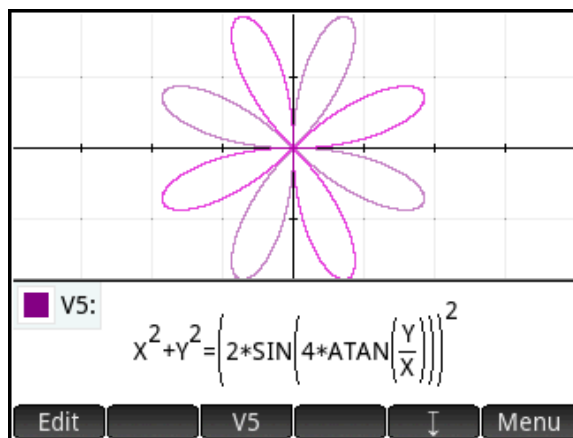
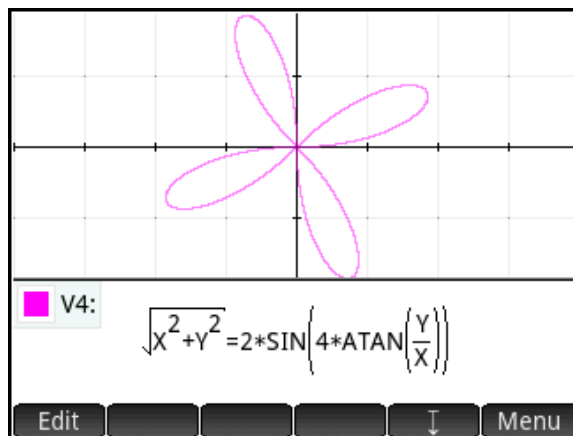


6. V3 is $\sqrt{X^2 + Y^2} = 2 \cdot \sin\left(4 \cdot a \tan\left(\frac{Y}{X}\right)\right)$.

The graph is a 4-petaled rose, but the polar equivalent has 8 petals. Here, the left side can only return non-negative values but the right side can return values in the interval $[-2, 2]$. So the values on the right side where $\sin(4 \cdot \text{atan}(Y/X))$ are in an interval (eg. $\pi/4 < Y/X < \pi/2$) where the results are negative are excluded.

7. If we square both sides, then the right side will only return non-negative results and so will the left side. This removes the restriction and plots the other 4 petals.

Note again that $X \neq 0$, so there is a hole at the center of our rose.



Extensions

1. Challenge the students to explain why V3 plots a double spiral instead of a single one.